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AMPLITUDE SATURATION OF SPHERICAL NONLINEAR WAVES

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1. INTRODUCTION

One of the significant effects of finite-amplitude sound propagation is that of saturation. At low intensity the received signal, at any point in the field, varies in proportion to the source amplitude. However, at higher intensity the amplitude at the receiver is reduced by nonlinear effects and, as the source strength is increased, eventually tends to a limiting value, known as the saturation amplitude.

Shooter, Muir & Blackstock (1974) conducted a theoretical and experimental study of the important case of saturation of spherically symmetric sound waves. Based on a heuristic argument, these authors give a theoretical expression for the saturation amplitude which agrees well with the experimental results. We refer the reader to this source for a discussion of the history of the problem.

The starting point for our discussion will be the spherical Burgers' equation

$$\frac{\partial u}{\partial r} - \frac{\gamma+1}{2a_0^2}\, u \frac{\partial u}{\partial \tau} + \frac{u}{r} = \frac{\Delta}{2a_0^3}\, \frac{\partial^2 u}{\partial \tau^2} \qquad (1.1)$$

where u is the particle velocity, a_0 the sound speed, Δ the diffusivity of sound and $\tau = t-(r-r_0)/a_0$ is a retarded time, based on the source radius r_0. The boundary condition on the source is that

$$u = u_0 \sin \omega\tau \qquad (1.2)$$

forming a well posed problem for the equation (1.1).

In a recent paper (Crighton and Scott (1979)) the problem (1.1) with (1.2) was investigated from the point of view of singular perturbation theory in the limit of amplitude saturation. The basic assumptions needed to derive equation (1.1) were discussed there. They are that : the source Mach number be small, that the source radius be large compared to the wavelength and that the quantity $\frac{\omega\Delta}{a_0^2}$ be small. The latter assumption says that diffusive effects are slow on the scale of one period.

2. FORMULATION OF THE REDUCED PROBLEM

We change variables to

$$R = R_0 \ln\left(\frac{r}{r_0}\right), \quad V = \frac{r u}{r_0 u_0}, \quad \theta = \omega\tau \qquad (2.1)$$

and define the two free parameters

$$R_0 = \frac{(\gamma+1)U_0\omega r_0}{2a_0^2}, \qquad \mathcal{E} = \frac{\omega\Delta}{(\gamma+1)U_0 a_0} \tag{2.2}$$

Equation (1.1) becomes

$$\frac{\partial V}{\partial R} - V\frac{\partial V}{\partial \theta} = \mathcal{E}e^{R/R_0}\frac{\partial^2 V}{\partial \theta^2} \tag{2.3}$$

while (1.2) reads

$$V(0,\theta) = \sin\theta. \tag{2.4}$$

Symmetry considerations show that the solution is periodic in θ and that

$$V(R,-\theta) = -V(R,\theta) \tag{2.5}$$

Increasing the source amplitude U_0 corresponds to decreasing $\mathcal{E}$ and increasing R_0, while maintaining the product $\mathcal{E}R_0$ fixed. We therefore examine (2.3) with (2.4) in the limit $\mathcal{E}\to 0$ and $\mathcal{E}R_0 = a$, which should correspond to saturation. Note the identity $a \equiv \alpha r_0$, where α is the usual small signal attenuation coefficient

$$\alpha = \frac{\Delta\omega^2}{2a_0^3} \tag{2.6}$$

This singular perturbation problem was discussed by Crighton and Scott (1979) and it is a consequence of their analysis that the saturated waveform is given by

$$V = \mathcal{E}F(\mathcal{E}R,\theta) \tag{2.7}$$

where the function $F(\xi,\theta)$ satisfied the reduced equation

$$\frac{\partial F}{\partial \xi} - F\frac{\partial F}{\partial \theta} = e^{\xi/a}\frac{\partial^2 F}{\partial \theta^2} \tag{2.8}$$

and the condition

$$\xi F \sim \pi\tanh\left\{\frac{\pi\theta}{2\xi}\right\} - \theta \tag{2.9}$$

as $\xi\to 0$, uniformly in $-\pi \leq \theta \leq \pi$. This problem depends only on the parameter $a \equiv \alpha r_0$. Equation (2.7) can be rewritten in terms of physical variables as

$$u = \frac{2a_0^2 a}{\omega(\gamma+1)r} F\left(a\ln\left(\frac{r}{r_0}\right), \omega\tau\right) \tag{2.10}$$

an exact expression for the saturated waveform within the limits of validity of (1.1) with (1.2). Of course, the question remains - how do we solve (2.8) with (2.9) ? In general this must be done numerically.

3. DISCUSSION OF THE REDUCED PROBLEM

The right-hand side of (2.9) represents the well know sawtooth profile of nonlinear acoustics. This evolves under the equation (2.8) and eventually goes into old-age, where the sound propagation is dominated by linear decay. The form of the old-age solution must be given by

$$F \sim C(a)\exp\left(-ae^{\xi/a}\right)\sin\theta \tag{3.1}$$

for $\xi \gg 1$. Which, in terms of physical variables, reads

$$u \sim D(\alpha r_0)\frac{a_0^2}{\omega(\gamma+1)r}\, e^{-\alpha r} \sin\omega\tau \qquad (3.2)$$

The old-age saturation amplitude is thus determined apart from the one unknown function $D(a)$ $(\equiv 2a\,C(a))$, whose evaluation represents the main goal of this study. In § 4 we will describe the numerical evaluation of $D(a)$, but for the moment we investigate the limits of small and large αr_0.

If we assume that αr_0 is large, then equation (2.8) becomes the Burgers' equation and we have the Fay solution

$$F = 2\sum_{n=1}^{\infty}\frac{\sin n\theta}{\sinh n\xi} \qquad (3.3)$$

which was shown to satisfy the condition (2.9) by Blackstock (1964). It then follows that

$$D(\alpha r_0) = 8\alpha r_0\, e^{\alpha r_0}. \qquad (3.4)$$

Not surprisingly, these results correspond to those for the saturation of plane waves - the curvature of the source is unimportant when αr_0 is large.

The situation is more complicated for αr_0 small. We wish to solve the problem (2.8) with (2.9) in the limit $a \to 0$. Initially we adopt the scaling

$$\bar{\xi} = \xi/a \qquad (3.5)$$

and the expansion

$$aF = \bar{F}_0(\bar{\xi},\theta) + o(1) \qquad (3.6)$$

to find that

$$\bar{F}_0 = \frac{\pi-\theta}{\bar{\xi}} \qquad (3.7)$$

for $0 < \theta < 2\pi$. That is, the main part of the wave remains a sawtooth in this region. The shock at $\theta = 0$ is described by the scaling

$$\bar{\theta} = \theta/a \qquad (3.8)$$

and the expansion

$$aF = \frac{\pi}{\bar{\xi}}\tanh\left\{\frac{\pi\bar{\theta}}{2\bar{\xi}e^{\bar{\xi}}}\right\} + o(1) \qquad (3.9)$$

The expansion in the shock is the first to break down. It does so because the next term in the expansion (3.9) becomes comparable with the first one when

$$a\bar{\xi}^2 e^{\bar{\xi}} = O(1) \qquad (3.10)$$

and so we define $\xi_1(a)$ by the relation

$$a\xi_1^2 e^{\xi_1} = 1 \qquad (3.11)$$

and adopt the coordinates

$$\hat{\xi} = \bar{\xi} - \bar{\xi}_1\,, \quad \hat{\theta} = \theta\xi_1 \qquad (3.12)$$

with the expansion

$$a\xi_1 F = \hat{F}_0(\hat{\xi},\hat{\theta}) + o(1) \qquad (3.13)$$

to obtain the equation

$$\frac{\partial\hat{F}_0}{\partial\hat{\xi}} - \hat{F}_0\frac{\partial\hat{F}_0}{\partial\hat{\theta}} = e^{\hat{\xi}}\frac{\partial^2\hat{F}_0}{\partial\hat{\theta}^2} \qquad (3.14)$$

and the matching condition

$$\hat{F}_0 \sim \pi \tanh\left\{\frac{\pi\hat{\theta}}{2e^{\hat{\xi}}}\right\} \tag{3.15}$$

as $\hat{\xi} \to -\infty$.

This region is very similar to one which occured in the study of spherical N-waves by Crighton and Scott (1979) and we refer the reader to this source for further discussion of the analogues of the above scalings and expansions. It was shown by the above authors that

$$\hat{F}_0 \sim \pi \operatorname{erf}\left(\tfrac{1}{2}\hat{\theta}e^{-\hat{\xi}/2}\right) \tag{3.16}$$

as $\hat{\xi} \to +\infty$, where erf denotes the error function. The "shock" is now governed by linear diffusion and so even though we cannot solve the problem for $\hat{F}_0$ we are able to continue the solution for larger ξ .

The next breakdown occurs because the shock width becomes of order unity when $a\exp(\bar{\xi})=O(1)$ and so we define

$$\xi_2 = |\ln| a \tag{3.17}$$

and then write

$$\tilde{\xi} = \bar{\xi} - \xi_2 \tag{3.18}$$

with the expansion

$$a\xi_2 F = \tilde{F}_0(\tilde{\xi},\theta) + o(1) \tag{3.19}$$

This results in the equation

$$\frac{\partial\tilde{F}_0}{\partial\tilde{\xi}} = e^{\tilde{\xi}}\frac{\partial^2\tilde{F}_0}{\partial\theta^2} \tag{3.20}$$

with the matching condition

$$\tilde{F}_0 \sim \pi \operatorname{erf}\left(\tfrac{1}{2}\theta e^{-\tilde{\xi}/2}\right) - \theta \tag{3.21}$$

as $\tilde{\xi} \to -\infty$, uniformly in $-\pi \le \theta \le \pi$. This is the region where the main wave expansion (3.6) breaks down and the shock and main wave merge together.

The solution to this problem is

$$\tilde{F}_0 = 2\sum_{n=1}^{\infty}\frac{\sin n\theta}{n}e^{-n^2e^{\tilde{\xi}}} \tag{3.22}$$

so that the old-age solution is

$$F \sim \frac{2}{a|\ln a|}e^{-ae^{\xi}/a}\sin\theta \tag{3.23}$$

and hence

$$D(\alpha r_0) = \frac{4}{|\ln \alpha r_0|} \quad . \tag{3.24}$$

The saturated waveform for small αr_0 is therefore as follows. It consists of a sawtooth containing shocks until we enter the region defined by $\tilde{\xi}$ and it then decays under linear effects. It will be noted that this is very similar to the situation envisaged by Shooter, et al. (1974) to obtain their theoretical model. In the region defined by $\bar{\xi}$ and $\bar{\theta}$ the shocks are quasi steady, this is not the case in the region defined by $\hat{\xi}$ and $\hat{\theta}$.

We have thus obtained the complete saturated waveform in the limits αr_0 small and large, apart from the shock breakdown region $(\hat{\xi},\hat{\theta})$ when αr_0 is small. In particular we have the limiting forms of $D(\alpha r_0)$.

4. GENERAL αr_0 : NUMERICAL SOLUTION

As mentioned before, Shooter, et al. (1974) gave a theory of saturation of spherical waves. Their results for the old-age saturation amplitude is of the form (3.2) provided that

$$D(\alpha r_0) = 4\Gamma(\alpha r_0)\, e^{\Gamma(\alpha r_0)} \tag{4.1}$$

where the function Γ is defined by

$$\Gamma(x)\, \ln\left(\frac{\Gamma(x)}{x}\right) = 1 \tag{4.2}$$

This equation for $D(\alpha r_0)$ agrees with the true result (3.24) in the limit of small αr_0 , but for large αr_0 their result becomes

$$D(\alpha r_0) = 4e\, \alpha r_0\, e^{\alpha r_0} \tag{4.3}$$

which differs from the true result (3.4) by the factor $e/2$. Indeed this discrepancy was pointed out by Shooter, et al. themselves.

A numerical integration of equation (2.8) with the condition (2.9) was undertaken by the present author to try and clarify this issue. The extremely singular nature of the initial conditions (2.9) presented stability problems, but, after some judicious transformations of the coordinates, it became possible to integrate the equation using a three-level, implicit, finite-difference scheme. The function $D(\alpha r_0)$, calculated in this way, is plotted in the diagram, along with the result of using equation (4.1).

A modified form of the heuristic formula (4.1) with (4.2) improves the agreement with the true form of $D(\alpha r_0)$. We modify the definition of $\Gamma(x)$ as follows

$$\left(\Gamma(x) - \left(1 - \frac{1}{\ln 2}\right) x\right) \ln\left(\frac{\Gamma(x)}{x}\right) = 1 \tag{4.4}$$

which leads to agreement with the true result in the two limits of αr_0 small and large. The result of this modification is plotted in the diagram and it will be seen that there is now good agreement with the true form over the entire range of values of αr_0 .

5. CONCLUSION

We have considered the equation for the saturated waveform which was previously derived using matched asymptotic expansion techniques. It has proved possible to solve it analytically in two limits, namely small and large αr_0 . In the small αr_0 case, the asymptotic structure supports the conjecture of Muir (1971) that the sawtooth region can be effectively matched into the old-age region. For intermediate values of αr_0 , the old-age saturation amplitude has been obtained by numerical integration of the equation and a modified form of the heuristic equation of Shooter, et al. (1974) leads to improved agreement with the numerical results.

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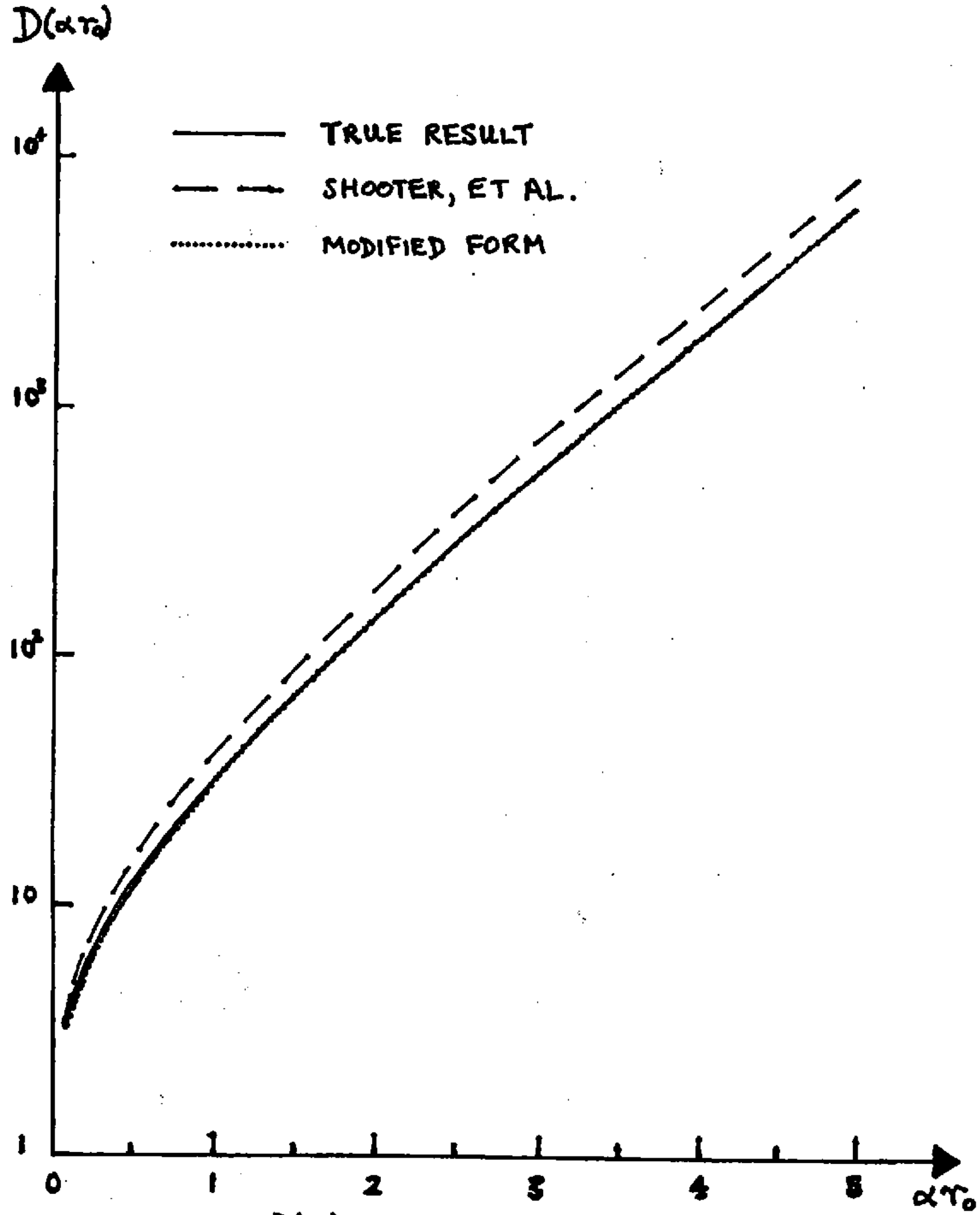

The function $D(\alpha\tau_0)$, occuring in the old-age saturation waveform (3.2)