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COMPUTATION OF RAY STATISTICS BEYOND A MULTI-SCALE DIFFUSER

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INTRODUCTION

A diversity of naturally occurring phenomena show a random geometric structure which appears similar over many decades of length scales. Examples are turbulence in fluids and in the atmosphere, coastlines, the "surface" of clouds and the trajectories of heavy particles which are scattered by light ones (Brownian motion). Mandelbrot [1] has coined the generic word "fractal" to describe such multi-scale "self-similar" objects. In this paper we shall be discussing the statistics and correlation properties of high-frequency waves which have encountered one such fractal object and we will assume the geometrical optics limit. As pointed out by Berry [2], the geometrical optics limit needs some care since, strictly speaking, a fractal is non-differentiable (though continuous). However, demanding that the eminating wavefront be once differentiable, ie that its slope is fractal, ensures that the geometrical optics limit is valid (Jakeman [3]).

For true fractals it is known (Berry [2]) that the intensity fluctuations of the scattered wave are weak whereas for the so-called "smooth" scatterers which produce wavefronts that are at least twice differentiable, there occurs very large fluctuations in intensity (infinite in the geometrical optics limit) when such wavefronts propagate in free space (see eg Berry [4]).

Wavefronts with fractal slopes, on the other hand, give rise to intensity fluctuations which can take all intermediate values depending on the so-called fractal dimension (Jakeman [5], Jakeman and Jefferson [6]) and thus represent an important classification of scatterers worthy of further study.

Although it is a relatively straightforward matter to devise a simple mathematical model to describe the statistics and correlation properties of a fractal wavefront, a determination of how these properties change when the wavefront propagates is fraught with mathematical difficulties. Indeed, this is not perculiar to fractals; very few statistical models yield analytical solutions for the statistics of the propagating wavefronts and even a computation of the second-moment of intensity (contrast) often proves laborious. One exception is the case when the slope of the wavefront is a so-called Brownian fractal for which the statistics and correlation properties may be derived exactly (Jakeman [3]). It is this exact solution which motivated the present work.

The intention was to simulate the Browninan fractal-slope wavefront on a computer and investigate the statistics and correlation properties of the eminating rays in the geometrical optics limit, comparing the results with the known analytic solution. We could then investigate the effects of finite size (inner and outer scale) and subsequently, with some confidence, apply the same computational techniques to other "non-Brownian" fractal-slope wavefronts for which the statistics are not known. The "discretised" Brownian fractal-slope model turned out to be an interesting problem in its own right which also yielded exact solutions that could be compared directly with the simulations and it is these results which will now be presented.

MODEL AND THEORY

If a plane wave is passed through a narrow region of turbulence or reflected from a rough surface then, assuming that the effect of the medium is to change only the phase of the wave, and taking the geometrical optics limit, an eminating wavefront may be described entirely by the random function $m(x)$, the gradient of the wavefront at point x.†

The model we shall take for $m(x)$ is that of a *Brownian fractal* which, when $m(x)$ is discretised, may be generated by the recursion relation

$$m_{n+1} = m_n + \sqrt{\Delta/l}\,\varepsilon_{n+1} \tag{1}$$

where the random numbers ε_i are δ-correlated and have unit variance; Δ is the step-length and l is a constant. Equation (1) is just the discretised Langevin equation for the one-dimensional Brownian motion of a free particle in a "frictionless" medium. The iterated solution to (1) is

$$m_n = \sqrt{\Delta/l}\sum_{i=0}^{n}\varepsilon_i \tag{2}$$

from which we easily deduce

$$\langle(m_{n+n'} - m_{n'})^2\rangle = x_n/l \tag{3}$$

where $x_n = n\Delta$ is the displacement and the angular brackets denote ensemble average. We note that m_n satisfies an "affine" scaling law, ie equation (3) remains invariant under the transformation $x_n \to kx_n$, $m_n \to k^{\frac{1}{2}}m_n$. This is shown in figure 1 where we have scaled the propagation direction linearly and the screen (x-direction) quadratically, with $l = \Delta = 1$ and $\varepsilon_i = \pm 1$. The ray diagrams look statistically equivalent in the sense that the rms change in slope over the screen widths is similar in all diagrams. We note, however, that there is a regular focussing of rays at various propagation distances, particularly in the first diagram, and that this is most pronounced at odd integral values of z. They occur because the rays have been generated on a uniform grid, $\Delta = 1$, and because $\varepsilon_i = \pm 1$ (giving an approximately gaussian distribution for m_n, since n is large). However, this is an inner-scale (discretisation) effect which is really inescapable since even if Δ were randomised and ε_n had some other statistics, eg Gaussian, there would still be "fuzzy" focussing. This regular (periodic) focussing does however disappear when the rays are viewed on a larger

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† We shall only consider a one-dimensional or "corrugated" wavefront which has constant phase in the y-direction and propagates in the z-direction.

scale, as expected (see figure 1D) and indeed the pattern becomes independent of the statistics of ε_n . (A consequence of the central-limit theorem.)

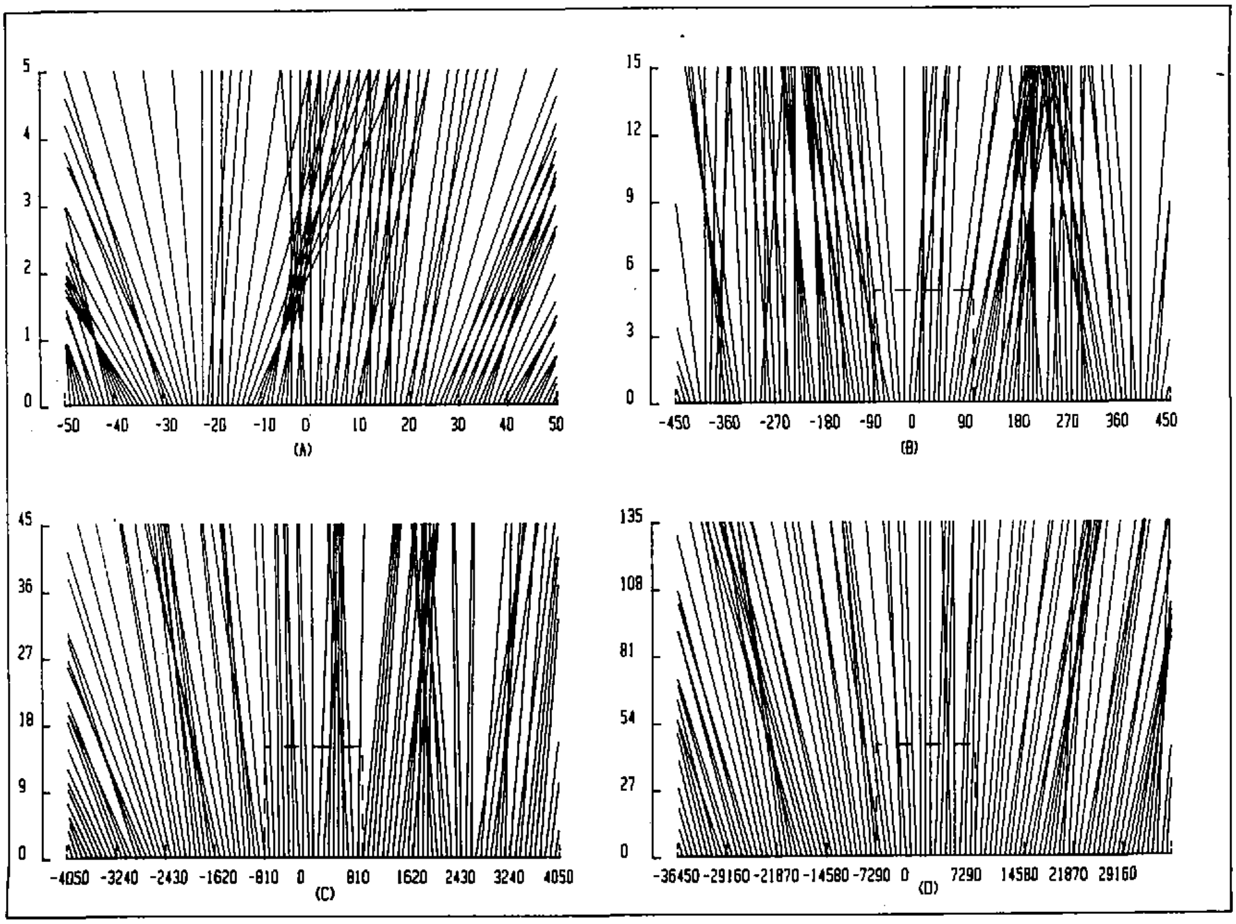

Figure 1. Ray diagrams under affine scaling transformations. Dashed lines denote boundary of previous diagram.

Using the fact that at odd integral z-values the focussed rays lie on a grid with a spacing of two, together with the Markovian property of m_i and combinatorial analysis, we have been able to show that the factorial moments and correlation function for the number of rays crossing the grid points are given by the following expressions [7].

$$k_m(z) \equiv \langle n_i(n_i-1) \ldots (n_i-m+1)\rangle = m! \sum_{l=1}^{\infty} \left[\frac{{}^{zl}C_{l(l+z)/2}}{2^{zl}} \right]^{m-1} \tag{4}$$

and

$$c_j(z) \equiv \langle n_i n_{i+j} \rangle = {\sum_{t=-\infty}^{\infty}}' \frac{{}^{|j+zt|}C_{(t+|j+zt|)/2}}{2^{|j+zt|}}, \quad j \geq 2 \tag{5}$$

where $n_i \equiv n_i(z)$ is the (random) number of rays crossing grid point i at distance z (odd-integral) and the prime on the summation in (5) means preclude all $t<0$ lying between $-j/(z-1)$ and $-j/(z+1)$. Equations (4) and (5) are exact. They are particularly simple for two cases; $z = 1$ and $z \to \infty$. At $z = 1$ (the first focussing region) $k_m(1) = m!$ and $c_j(1) = 2$, showing that the n_i are δ-correlated. It is straightforward to invert the factorial moments to give a probability distribution $P_n(1) = (\frac{1}{2})^{n+1}$. These results can, in fact, be obtained much more simply by noting that only adjacent rays can intersect at the same grid point at $z = 1$ (see figure 1A) and that the probability of no rays passing through is exactly $\frac{1}{2}$. Thus $P_n = (\frac{1}{2})^n.\frac{1}{2}$ where, for $n > 0$, the first factor is the probability that n adjacent rays will converge onto the grid point whilst the remaining $\frac{1}{2}$ is the probability that the next ray will not. Once the chain of adjacent converging rays is broken no other rays can ever converge to the same point at $z = 1$ and thus the number of rays at different sites are independent (δ-correlated). Thus we have a very simple demonstration of how the highly correlated Gauss-Markov process m_i is transformed into a δ-correlated process n_i with non-gaussian statistics.

The δ-correlation occurs only at $z = 1$. As z increases, near-neighbour sites become increasingly correlated and we can in fact show that for large z the correlation length increases quadratically with z, an effect which is seen clearly on the ray-diagrams. Using Stirling's formulae to approximate the combinatorial factors in equations (4) and (5) it can be shown [7] that for large z (odd-integral) $k_m(z) \to m!(2)^{m-1}$ and $c_j(z) \to 2[1 + exp(-2j/z^2)]$. Again $k_m(z)$ can be inverted to yield $P_{n>0}(z) = (2/3)^{n+1}/4$ with $P_0(z) = 2/3$. For $j \ll z^2$ ray numbers on different sites are almost *fully correlated* and using this fact it is not difficult to show that the mean *ray density* over a length $2j$, ie $R_j = (n_1+n_2+ \ldots\ldots n_j)/2j$ has moments $\langle R_j^m \rangle \simeq m!$

More precisely, we can show rigorously that

$$\langle R_j^m \rangle \xrightarrow[\substack{j,z \to \infty \\ j/z^2 \to 0}]{} m!$$

and thus, in this limit, $P(R) = e^{-R}$, a result obtained by Jakeman [3] directly for the continuum limit.

SIMULATION

To check the analysis outlined in section 2, we have performed computer simulations generating several hundred Brownian walks each with 100,000 rays using equation (1). Theoretical plots of the first six factorial moments are plotted in figure 2 (solid lines, cf equation (4)) together with the simulated values and theoretical asymptotes ($z \to \infty$, dashed lines).

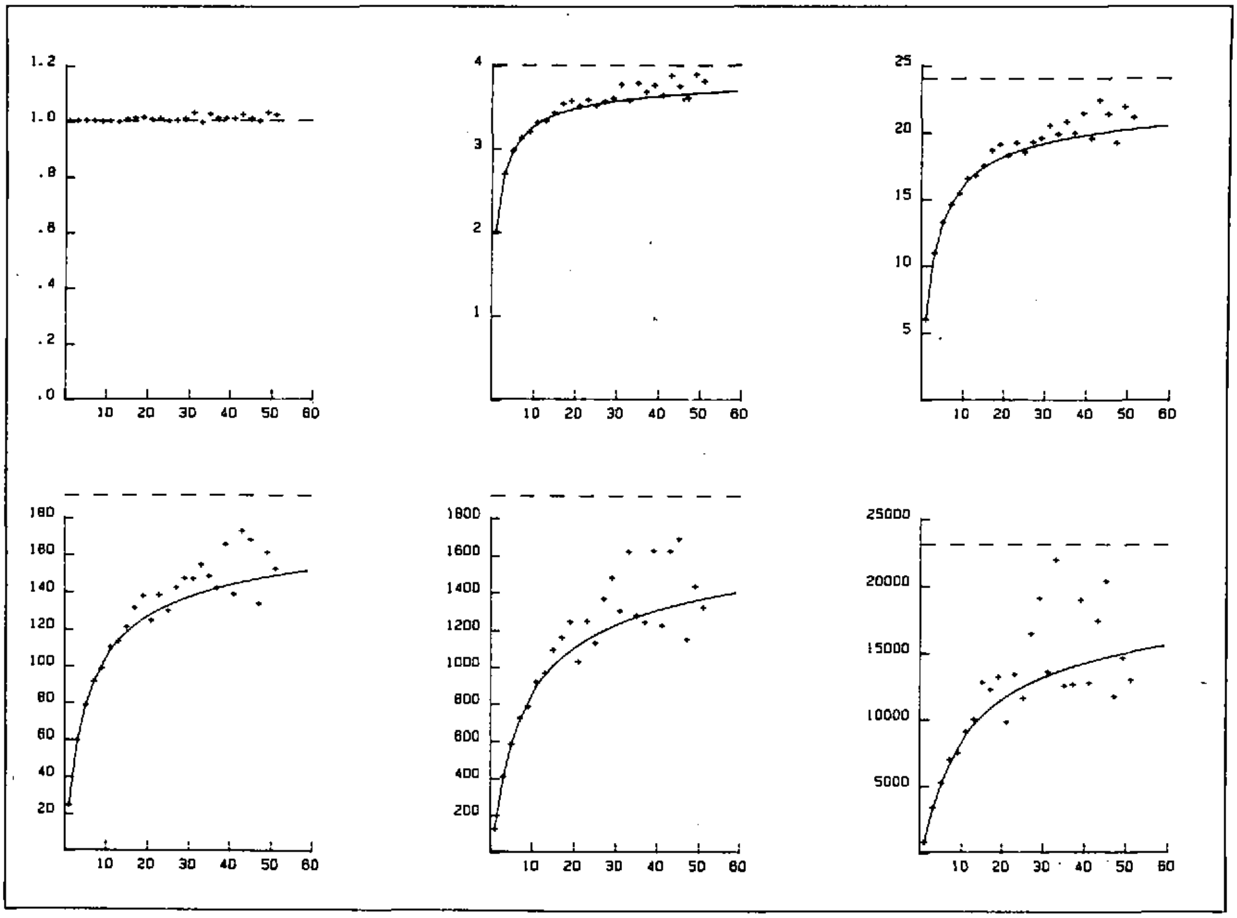

Figure 2. First six factorial moments, —— theory, --- asymptotic values, + simulation (100 runs of 100,000 rays).

These plots show good agreement between theory and computer "experiment" but it should be noted that convergence to the asymptotic values is quite slow. This is purely an inner-scale (discretisation) effect showing that, in general, care should be exercised when attempting to extract asymptotic statistics from simulations of wave-fields which have passed through multi-scale refracting media.

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