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PRONY AND FOURIER METHODS APPLIED TO DUCT ACOUSTICS

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INTRODUCTION

The measurement of plane wave acoustic fields in flow ducts require a signal analysis technique that represents the signal by a sum of sine wave components with high precision. The most widely used technique is the Fourier Transform in its various forms. For discrete component digital signals the Discrete Fourier Series can be written as

$$f(k\Delta) = \sum_{j=0}^{M-1} C_j e^{i2\pi kjf_o\Delta}; \quad k = 0,1,\ldots,M-1 \qquad (1)$$

where $f_o=1/T$ and T is the period of the signal, Δ is the digitisation period, M is the number of samples of the signal. The application of Fourier analysis to problems where the wave components are not in strict harmonic sequence will generate erroneous results. For example, the analysis of the quasi-periodic signals found in combustion engine exhaust pipes.

Alternatively, a method based on the Prony Series defined by

$$f(k\Delta) = \sum_{j=0}^{2N-1} C_j e^{s_j k\Delta}; \quad k = 0,1,\ldots,M-1 \qquad (2)$$

where $s_j=\sigma_j+i2\pi f_j$, can be used with advantage in some cases for signal component identification. N is the number of components of the real function f, while σ_j and f_j are respectively the damping factor and the frequency of the jth component.

THE METHODS

The Fourier representation is designed for signals with a broadband spectrum or of repetitive nature. The Fourier Series (1) fits a set of harmonically related components to the signal $f(k\Delta)$. The period of the signal has to be accurately defined as it determines the 'fundamental' frequency f_o. In practice, if the M samples comprise n periods

$$f_o = n/M\Delta. \qquad (3)$$

This relation shows the synchronisation of the sampling frequency $fs=1/\Delta$ with f_o. The frequencies f_s are normally preset in the analogue-to-digital converter. Consequently, if the driving frequencies are not adjusted to fs for a correct detection, a high sampling rate fs and a large number of samples have to be used for a best estimation. Relation (3) also defines the frequency resolution. Again, a fine discrimination can only be guaranteed by high values of fs and M. The FFT normally requires M to be a high power of two.

The Prony Series is an ideal representation of discrete component signals [1]. Comparing the Prony Series (2) to the Fourier Series (1), one can see that the exponents for the Prony Series are complex and need not be harmonically related. No synchronisation of sampling and acoustic frequencies is necessary, while a minimum of 4N samples is required. This means that very short data lengths can be analysed using the Prony method.

THE PRONY SERIES ALGORITHM

The representation of a signal by a series of complex exponentials was introduced by Prony as early as 1795 [2]. The application to digital signals has only been investigated in the 1960's [3], [4]. For the evaluation of the exponents s_j in the non-linear set of equations (2), Prony introduced a new variable α_j as the coefficients of a 2Nth order polynomial in x whose roots $x_j = \exp(s_j\Delta)$. A matrix and a polynomial equation have to be solved sequentially.

(i) Solve:

$$\sum_{k=0}^{2N-1} f((kp+nq)\Delta)\alpha_k = f((2Np+nq)\Delta) \qquad (4)$$

$$n = 0,1,\ldots,M-1$$
$$p,q \text{ integers}$$

to determine the coefficients α_k; p represents the separation among the samples taken for coefficients of each equation, q is the separation among the first samples of the sequential equations.

(ii) With $\alpha_{2N}=1$, find the roots of

$$\sum_{k=0}^{2N} \alpha_k x^k = 0. \qquad (5)$$

The s_j can be calculated from the solutions x_j

$$s_j = \frac{1}{P\Delta}\ln(x_j).$$

Finally, the evaluation of the complex amplitudes A_j is a fitting problem:

(iii)

$$\sum_{j=0}^{2N-1} A_j x_j^k = f(k\Delta); \quad k = 0,1,\ldots,M-1 \qquad (6)$$

The introduction of the parameters p and q is to obviate the fact that the matrix equation (4) becomes increasingly ill-conditioned as the digitisation period decreases [3],[5]. The time interval spanned by each one of equations (4) is $Np\Delta$. This should be as long as possible. This calls for either a large number N of analysis components or a low sampling rate $1/\Delta$. The resulting effective sampling rate is

$$fseff = 1/p\Delta.$$

The increase on p is limited in the analysis of noisy signals by the 'aliasing' of the noise high frequencies. The effect of q is to extend the total number of samples covered by the matrix equation (4), for the same memory allocation.

The Prony program, using least squares for the solution of (4) and (6) when the number of samples $M>2N(p+q)$, was found to be slower than a standard FFT Program. Its speed was comparable with a Discrete Fourier Series Program (DFS), provided N was less than 9.

APPLICATIONS OF THE METHODS

The methods were applied both to signals extracted from an experimental rig and to signals generated in the computer (PDP 11/50).
The estimation of the damping factors σ_j has not presently been exploited. But it was observed that the algorithm picked up any variation in the envelope noise etc.

For noise free signals the three Programs (FFT, DFS and Prony) compared well, when the synchronisation condition (3) was satisfied. For signals with multiple non-harmonic discrete components, the Prony algorithm was invariably superior. The following table shows the results of the analysis of a signal composed of three pairs of undamped components by the Prony Program.

SIGNAL :	fs(Hz)	Anal.	f(Hz)	Amp	θ(rad)
		N = 8	55.979	9.510	4.712
	8k	M = 200	501.395	6.505	4.712
sum of		p=q=1	700.389	6.502	4.712
			54.969	4.943	4.775
55.0 Hz,Amp-5.0		N = 12	57.002	4.578	4.644
57.0 Hz,Amp-4.5	8k	M = 300	500.010	2.013	4.698
500.0 Hz,Amp-2.0		p = 2	502.001	4.487	4.719
		q = 2	700.011	2.333	4.753
502.0 Hz,Amp-4.5			701.054	4.178	4.640
			55.008	5.044	4.713
700.0 Hz,Amp-2.5		N = 8	57.012	3.956	4.712
701.0 Hz,Amp-4.0	4k	M = 272	500.005	2.010	4.714
		p = 2	502.002	4.490	4.712
		q = 1	699.987	2.431	4.716
			700.991	4,069	4.710

The table shows that by changing the parameters of analysis the components can eventually be correctly discriminated. When this does not occur, as in the first estimate, the energy of the two components f_1 and $f_2 = f_1+\Delta f$ was concentrated in the frequency in between $f_1+\Delta f/2$. Note that a lower sampling rate f_s improves the resolution.

When the signals were affected by ambient noise, it was in general observed that the nature of the noise influences the results more than the S/N ratio. The worst cases corresponded to broadband noise. The algorithm picked up noise components all over the bandwidth, impoverishing the resolution. The next table shows the analysis by the Prony Program of a signal embedded in broadband background noise in three different ways.

SIGNAL :	Anal.	f(hz)	Amp	θ(rad)
	N = 15	490.001	10.001	4.712
(1) Pure signal (S/N=∞)	M = 300	499.999	12.499	4.713
	p=q=1	699.999	14.999	4.712
(2) Signal (1) plus	N=14,p=1	497.786	27.075	4.299
broadband noise(S/N=23dB)	M=296,q=1	699.970	14.975	4.724
(3) Signal (1) plus the	N = 14	490.480	11.066	4.799
broadband noise filtered	M = 296	500.355	11.516	4.630
in the 400-600 Hz band	p=q=1	700.008	14.998	4.712
(4) Signal (3) but noise	N = 13	490.289	12.782	5.296
level increased to	M = 292	501.090	13.846	4.170
equal S/N=23dB of (1)	p=2,q=1	699.998	15.007	4.713

The table shows that the lower the S/N ratio, the coarser the resolution. But, for the same S/N ratio the components can be separated when the noise is narrowband. The Fourier estimates were less affected by the presence of noise than the Prony ones. This is explained by the much larger lengths of data input to the Fourier techniques. Several strategies were followed to reduce the effects of noise on the estimations. The averaging technique was seen to perform equally well with Fourier and Prony techniques. But, it requires the knowledge of the period $\hat{T}$ of the signal, i.e. the synchronisation stated in (3). Filtering techniques were also successfully employed. The characteristics of the filter had to be made as steep as possible, for maximum efficiency.

CONCLUSIONS

The Prony analysis produced correct estimations when compared to the ones obtained by Fourier methods. For signals embedded in broadband ambient noise the Prony method required for some cases a further processing, either by changing the parameters or by using a multiple procedure (averaging or filtering prior to the analysis). When Fourier methods cannot be employed, as for signals with a set of non-harmonic components or very short data lengths, the Prony algorithm can provide good estimations.

The use of the Prony analysis brings a great economy both in data length and in experimentation time. The latter is due to the relaxation of the requirement for the synchronisation of the driving and sampling frequencies.

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