SEQUENTIAL PROCESSING OF MULTIPLE DATA DOMAINS USING SVD

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ABSTRACT

We describe how the utilisation of a sequential processing strategy for data received at a limited number of spatially distributed sampling points enables us to deal with information from multiple domains in an efficient manner. However, we show by means of a simple example that the accuracy of parameter estimation in each succeeding domain may be sensitive to the actual sequence adopted.

1. INTRODUCTION

A large number of modern antenna array signal processing techniques, such as MUSIC [1], concentrate on the problem of recovering linearly filtered signals from data disturbed by Gaussian observation noise. They often make use of the Karhunen Loève expansion of data in terms of a complete orthonormal set of basis functions [2] and assume that sources of signal are independent. These algorithms are usually applied to the case of multiple snapshots of data received at a limited number of spatially distributed sampling points. In this case, the ability of such algorithms to discriminate sources which appear to be highly correlated in the spatial domain derives from decorrelation which is assumed to have taken place in the time domain. Clearly it is possible to conceive alternative measurements which might detect decorrelation in additional domains, such as polarisation or a second spatial direction, with consequent gains in the ablity to discriminate multiple sources. However, as the number of domains of data to be processed increases, so does the time taken to search them exhaustively [3]. We have therefore made use of a sequential processing strategy [4] which uses the results of target discrimination in one domain as support for a less exhaustive search in each subsequent domain through the action of a matrix pseudo-inverse.

In section 2 we outline our approach, using the linear prediction algorithm of Kumaresan and Tufts [5] to process the data at each stage. In section 3 we present the results of computer simulations which demonstrate that, even for the simplest case of a single snapshot of data taken from a planar array of sensors, the order in which the two spatial domains are investigated can sometimes have a significant impact on the limiting accuracy with which source parameters may be estimated.

2. THE ALGORITHM

2.1. Singular Value Decomposition (SVD).

The singular value decomposition of a general matrix, $\underline{D}$, of dimensions $(m \times n)$, with $m \leq n$, may be written

$$\underline{D} = \underline{U}\,\underline{S}\,\underline{V}^H \qquad (1)$$

where the superscript H denotes the hermitian (complex conjugate) transpose, and

$$\underline{V}^H\,\underline{V} = \underline{U}\,\underline{U}^H = \underline{I}_m \quad . \qquad (2)$$

$\underline{S}$ is a diagonal matrix containing the non-negative square roots of the eigenvalues of $\underline{D}\,\underline{D}^H$ conventionally arranged in non-increasing order. The columns of $\underline{V}$ are the m orthonormalised eigenvectors corresponding to the m largest eigenvalues of $\underline{D}^H\underline{D}$ (there also exist n-m eigenvectors in this space which are associated with the remaining n−m zero eigenvalues of $\underline{D}^H\underline{D}$), and the columns of $\underline{U}$ are the orthonormalised eigenvectors of $\underline{D}\,\underline{D}^H$. The columns of $\underline{U}$ and $\underline{V}$ are respectively known as the left and right hand singular vectors of $\underline{D}$, and the elements $\sigma_1,\ldots,\sigma_m$ of the matrix $\underline{S}$ are called the singular values of $\underline{D}$. $\underline{I}_m$ is the $(m \times m)$ identity matrix.

The number of non-zero singular values defines the rank of the matrix $\underline{D}$. If there should be a number of the singular values which are very small in relation to others ($\sigma_1 \gg \sigma_m$), these may be set to zero in equation 1 in order to construct a reduced rank approximation of $\underline{D}$. This is a common technique for reducing the degree of ill-conditioning in matrix inversion problems [6].

2.2. Signal Subspace Algorithms.
Taking the simple example of a two domain problem (space − time or space − space), we assume that data collected from a spatially distributed array of m sensors subject to additive white Gaussian receiver noise may be represented by the equation

$$\underline{D} = \underline{A}\,\underline{F} + \underline{N} \qquad (3)$$

where $\underline{A}$ is a matrix containing the array response vectors, $\mathbf{a}(\theta)$, corresponding to the directions of the signal sources, $\underline{F}$ is the matrix describing the behaviour of the signal sources over the multi-domain observation aperture, and $\underline{N}$ is the matrix of noise samples taken from a Gaussian distribution. For a large number of samples of data, we assume that signals and noise are totally decorrelated.

A suitable set of Karhunen Loève basis functions is determined from either the left or right hand singular vectors of $\underline{D}$. If the total number of independent signal sources is less than the smaller dimension of $\underline{D}$, examination of the singular values allows us to interpret the complete orthonormal basis as describing a pair of mutually orthogonal subspaces, representing components containing predominantly signal (σ large) or noise (σ equal to the smallest singular value, σ_m). The MUSIC algorithm [1] makes use of the singular vectors in the noise subspace, projecting them against the array response vectors in order to derive angles of arrival or signal frequencies, θ_i, from the positions of minima or nulls in the plot of

$$|P_{MU}(\theta)| = |\mathbf{a}(\theta)^H\,\underline{U}_n\,\underline{U}_n^H\,\mathbf{a}(\theta)| \qquad (4)$$

where the subscript n denotes the noise subspace.

In the case of a short sample of data, the assumption that signals and noise are totally uncorrelated is generally invalid. In these cases linear predictive modelling approaches, stabilised by rank reduction, have been found to be more successful for processing data from a regularly spaced array lattice. In our own experiments, the linear prediction algorithm of Kumaresan and Tufts [5] has been found to give superior performance to MUSIC in simulations of linear and rectangular-grid arrays at the limits of their discrimination performance.

One possible interpretation of this algorithm starts from the equation for a linear prediction error filter acting on the data matrix itself

$$\underline{D}^H\,\mathbf{w} = \mathbf{0} \qquad (5)$$

where $\mathbf{w}$ is the required filter, and $\mathbf{0}$ is the null vector. In order to reduce the sensitivity of this filter to the noise content of $\underline{D}$ we make use of the singular value decomposition to form a reduced rank approximation of $\underline{D}$. However, as a result of taking very few samples of data, the singular values of $\underline{D}$ do not exhibit a clear distinction between noise and signal dominated subspaces. Utilising one of the numerous algorithms which now exist for deriving an approximate partitioning [7] enables us to attempt the formation of the required reduced rank estimate of $\underline{D}$, written in terms of the signal subspace vectors alone. Clearly the right hand vectors and the singular values of $\underline{D}$ may be omitted, allowing us to re-write (5) in the simple form

$$\underline{U}_S^H \, \mathbf{w} = \mathbf{0} \quad . \tag{6}$$

Setting the first element of **w** to be unity, the solution, written in terms of the noise subspace singular vectors, is

$$\mathbf{w} = \underline{U}_n \, \mathbf{u}_n \; / \; (\mathbf{u}_n \, \mathbf{u}_n^H) \tag{7}$$

where $\mathbf{u}_n$ is the first row of the noise partition singular vector matrix, $\underline{U}_n$.

Since **w** has been derived as the vector which cancels signals present in the data, the signal locations may be determined as the null positions in the space of possible solutions, given by the function

$$|P_{KT}(\theta)| = |\mathbf{a}(\theta)^H \, \mathbf{w} \, \mathbf{w}^H \, \mathbf{a}(\theta)| \quad . \tag{8}$$

The first direction finding or frequency estimation stage of the algorithm is now complete. By merely choosing to use either the left or the right hand set of singular vectors, we have chosen which of the two domains in this simple two-domain problem to work with first. The result of this stage of processing may now be used to provide support for further processing in the second domain.

The linking stage of our sequential approach is the formation of the pseudo-inverse of a matrix, $\underline{A}_S$, which contains only those response vectors from the first domain which correspond to the positions of nulls in the result given by equation 8. If we assume that as the first stage of a space - time problem we have found the directions of arrival of possible signal sources, the action of pre-multiplying the data matrix, $\underline{D}$, by the pseudo-inverse, $\underline{A}_S^{-1}$, is to extract a number of time series of data, each of which corresponds to one of the directions defined by the columns of $\underline{A}_S$, and is a row of the matrix

$$\underline{T} = \underline{A}_S^{-1} \, \underline{D} = \underline{A}_S^H \, (\underline{A}_S \, \underline{A}_S^H)^{-1} \underline{D} \quad . \tag{9}$$

Computation of the matrix $\underline{T}\,\underline{T}^H$ enables us to measure the total cross and auto powers in the recognised directions, and potentially to reduce the number of vectors in $\underline{A}_S$ should any prove to have been generated spuriously, since the power in those directions may appear at the noise level.

Each of the time series may now be investigated in turn or in parallel, utilising a sliding window in the time domain in order to construct a set of new data matrices. Thus if the data vector is

$$\mathbf{t} = (t_1 \; t_2 \; t_3 \; \ldots \; t_n) \tag{10}$$

and this is examined with an r-element window, we can build up the following matrix:

$$\underline{W} = \begin{matrix} t_1 & t_2 & \ldots & t_r \\ t_2 & t_3 & \ldots & t_{r+1} \\ .. & .. & \ldots & .. \\ t_{n+1-r} & \ldots & \ldots & t_n \end{matrix} \tag{11}$$

Each of the matrices $\underline{W}$ may be analysed in turn or in parallel by the Kumaresan and Tufts algorithm and a characterisation of the expected signals in the second domain in the form of an $\underline{A}$ matrix. The extension to further domains is straightforward [3].

3. COMPUTER SIMULATIONS

We have carried out numerous simulations of the sequential algorithm acting on data generated from scenarios involving either two or three domains [3,4]. We present here typical results obtained from modelling a single snapshot of data from a 20×20 planar array of antenna elements arranged on a rectilinear grid. Angular locations of targets are measured in generalised units of 'cycles' of phase over the antenna along the appropriate axis. One cycle corresponds to a conventional Fourier beamwidth. Our far field signal sources are separated by only 0.15 cycles in azimuth, but by 0.9 cycles in elevation. The signal to noise ratio at each antenna element is 6dB, noise samples being independently drawn from a zero mean Gaussian distribution and added with uniform weighting to the array output.

Examination of the singular values of the data matrix indicates a probable signal subspace dimension of two. This estimate is supported by analysis of the eigen spectrum by the MDL processor described by Wax and Kailath [7]. However, investigation of the azimuth domain using the Kumaresan and Tufts algorithm produces only a single null in $|P_{KT}(\theta)|$, approximately centrally located between the two sources (Fig 1a). Fig 1b shows the same result interpreted in terms of the

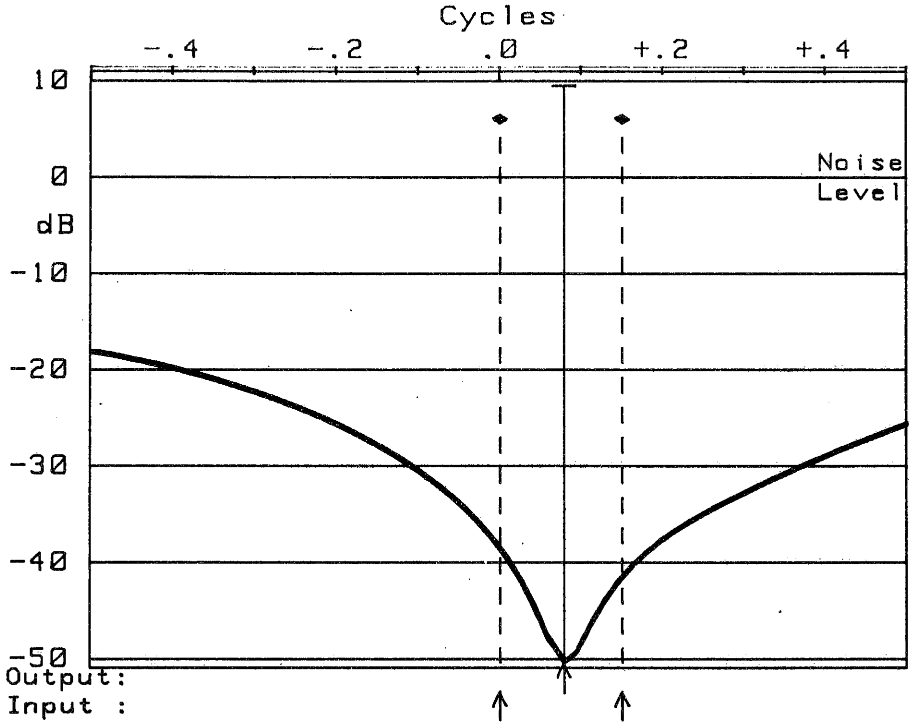

Fig 1a. Azimuth analysis of two emitter problem using Kumaresan and Tufts algorithm.

zeros of a z-plane polynomial with coefficients given by the elements of **w** [5]. A single zero, corresponding to the null in Fig 1a, is placed close to the unit circle. Subsequently, a moving window analysis of the data series in elevation associated with the single azimuth identified reveals the elevation angles of the two sources with only slight error (Fig 2). Thus, although the sequential approach has isolated two sources, they are not separately located in azimuth. If, instead of the sequence chosen above, we choose to process the data using the singular vectors in the elevation domain as the first stage, the two angles in this domain are determined to within a fraction of a cycle (Fig 3). A spurious null in this plot is distinguished and rejected from further examination on the basis of the low power

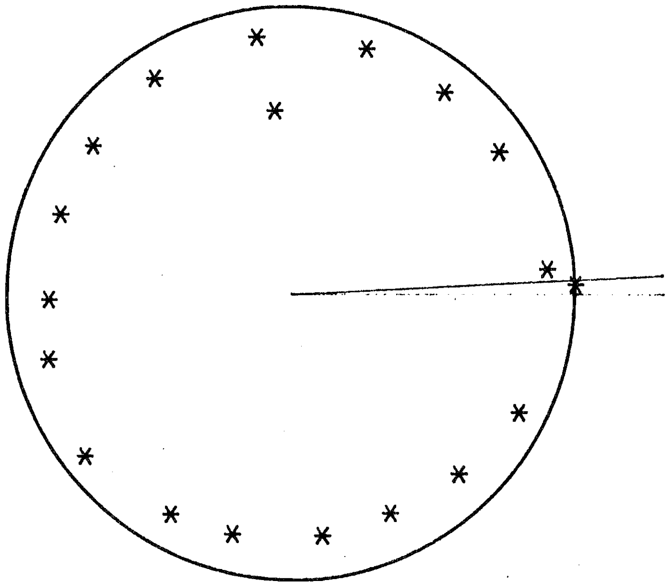

Fig 1b. Positions of zeroes in z-plane for azimuth problem.

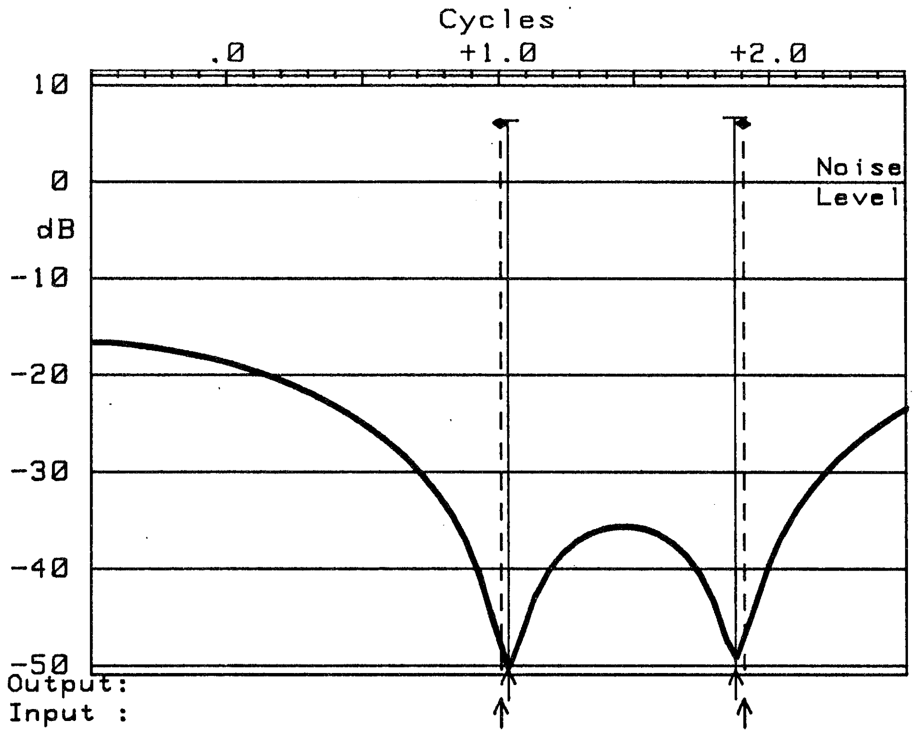

Fig 2. Discrimination of signals in elevation using result of Fig 1a.

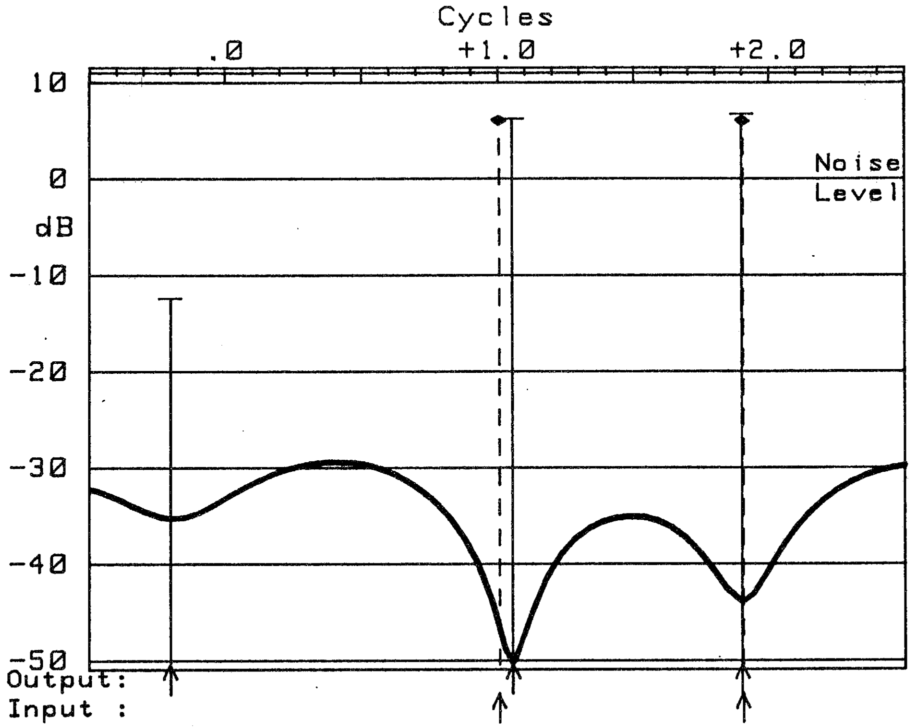

Fig 3. Analysis of elevation domain as first stage of algorithm.

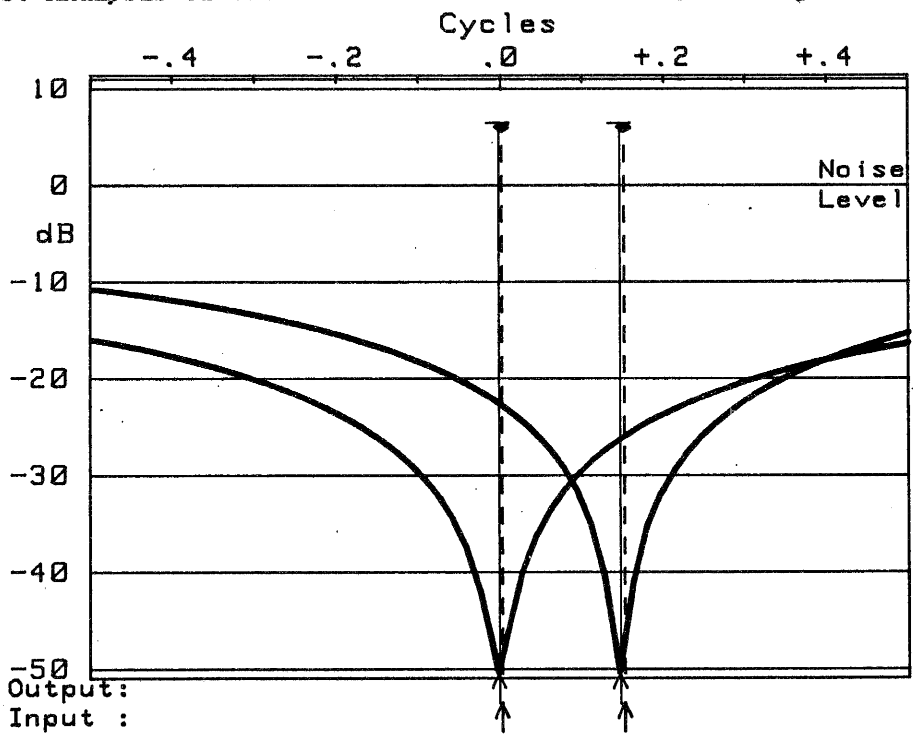

Fig 4. Analysis of azimuth as second stage of sequential algorithm.

estimated in that direction. Subsequent analysis of each of the data series associated with the two main frequencies in turn enables the locations of both emitters in azimuth to be found independently and accurately (Fig 4). Therefore, using the same realisation of data, but ordering our sequence of processing differently, we obtain a more accurate reconstruction of the objects in one case than the other.

4. CONCLUSIONS

We have presented a computationally efficient extension to the normal signal subspace algorithm which allows a detailed investigation of multiple domains of data through the utilisation of the results at each stage as prior knowledge for a reduced search at each succeeding stage. Although our results have demonstrated its application to a simple two domain problem, the extension to multiple domains is clear. However, our results suggest that the algorithm should be used with care, perhaps serving as an initial coarse processing stage before a more detailed investigation of the regions identified to be of interest.

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