# EVALUATION OF DIFFERENT COMPUTATIONAL STRATEGIES FOR ACOUSTIC FINITE ELEMENT MODELLING

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## 1. ABSTRACT

The finite element method has, for a long time, been applied to acoustic problems in the "low" frequency range. Usually response evaluation relies on either direct response computation or modal superposition. The modal approach is characterized by the selection of some reduced basis into which the original problem is projected and solved.

An alternate non–modal approach is proposed here for acoustic finite element models with complex impedance boundary conditions. The new procedure relies on generation of complex Ritz vectors. This basis is shown to have specific advantages over conventional modal basis : Ritz vectors can be generated easily and for the same precision level, the number of Ritz vectors can be shown to be less than the number of true eigenvectors.

Examples are presented in order to demonstrate the efficiency of the proposed technique which has been implemented into SYSNOISE program [15] (General purpose program for acoustic modeling).

## 2. INTRODUCTION

Application of the finite element methods to acoustic problems is not new. In fact, Helmholtz equation can be considered as a particular case of Navier equation so that it has been recognized, for a long time, that interior acoustic problems can be solved using structural FE programs [1]. Acoustic problems are however characterized by some specific items. They are formulated in complex terms (impedance boundary conditions, excitations not in phase, ...) so that practical treatment must rely on programs allowing for complex variables computation and specific post–processing facilities (spectral diagrams, directivity plots, etc...)

This requirement is even stronger for exterior (unbounded) radiation problems where finite element approach is not so well suited. In such cases, boundary element methods can be used [2].

In this paper, attention will be devoted to a new non–modal approach allowing to handle effciently acoustic finite element models.

## 3. ACOUSTIC FINITE ELEMENT MODEL

Acoustic problems are governed by the wave equation :

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \text{ in V} \tag{1}$$

where $c$ is the sound speed and $\psi$ the velocity potential related to pressure $p$ and velocity $v$ through relation

$$p = -\varrho \frac{\partial \psi}{\partial t} = -\varrho c^2 \, div \, u \tag{2}$$

$$v = \nabla \psi \tag{3}$$

where $u$ and $\varrho$ represent fluid displacement and density respectively.

Alternatively equation (1) may be formulated in terms of $p$ :

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \tag{4}$$

The problem is usually subjected to some boundary conditions (supplemented with appropriate initial conditions not described here).

$$p = \bar{p} \qquad \text{on } S_1 \tag{5}$$

$$\frac{\partial p}{\partial n} = -\varrho\, \bar{a}_n \qquad \text{on } S_2 \tag{6}$$

where $\bar{p}$, $\bar{a}_n$ denote fixed pressure and normal acceleration respectively.

Finite element modeling of conservative dynamical systems relies on Hamilton's principle. This principle states that between two instants of time $t_0$ and $t_1$, the motion proceeds so that the integral

$$J = \int_{t_0}^{t_1} L dt \tag{7}$$

is stationary for all pressures which satisfy boundary condition (5) and which coincides with the actual pressure of the system at $t_0$ and $t_1$. The so-called "Lagrangian" $L$ is given by

$$L = T - U_i - V_e$$

where $T$ is the kinetic energy, $U_i$ the strain energy and $V_e$ the external potential energy

$$T = \frac{1}{2}\int_V \varrho\; v.v\; dV \tag{8}$$

$$U_i = \frac{1}{2}\int_V \varrho c^2\, (div\; u)^2\, dV \tag{9}$$

$$V_e = \int_{S_2} p\; u_n\, dS \tag{10}$$

In most of the cases, additional impedance or admittance boundary conditions are present which relate normal velocity and pressure

$$p = Z_n\, v_n \;\; \text{on } S_3 \tag{11}$$

or

$$v_n = A_n\, p \;\; \text{on } S_3 \tag{12}$$

where $Z_n$ and $A_n$ are normal impedance and admittance values respectively.

The usual impedance boundary conditions lead to the introduction of dissipative effects. A special variational formalism for dissipative systems has been introduced by Morse and Feshbach [3], and has since been used by Gladwell [4, 5] and Craggs [6] for damped acoustic structural problems. The procedure involves the use of an adjoint system in which energy accrues as it is being dissipated in the physical system. The extended system is thus conservative and the equations of motion can be written in Hamilton's canonical form.

First variation of the resulting functional with respect to the adjoint variables gives rise to the equation governing the physical problem while the variation with respect to the original variables yields those governing the adjoint problem. Concentrating on the original problem, it can be shown that final discrete system takes the following form [7]

$$M\,\ddot{p} + C\,\dot{p} + K\,p = f \tag{13}$$

where the forcing vector $f$ is dependent on normal accelerations prescribed on $S_2$. Matrices $K$, $M$ and $C$ are given by

$$K = \sum_e \int_{V^e} B^T B \; dV^e \tag{14}$$

$$M = \sum_e \int_{V^e} \frac{N^T N}{c^2} \; dV^e \tag{15}$$

$$C = \sum_e \int_{S_3^e} \varrho \, A_n \, N^T N \; dS_3^e \tag{16}$$

while $p$ denotes now the vector of nodal pressures. Above expressions contain usual shape functions matrices $N$ and their cartesian derivatives $B$.

## 4. CONVENTIONAL SOLUTION PROCEDURES IN THE FREQUENCY DOMAIN

Usually acoustic problems are solved in the frequency domain assuming a time dependence like $e^{i\omega t}$, so that system (13) can be formulated as :

$$(K + i\omega C - \omega^2 M)\; p = f \tag{17}$$

where $p$ and $f$ denote now pressure and excitation amplitudes.

Various solution procedures can be used to get nodal pressures from (17). The ususal approach consists to solve directly (17) at discrete frequencies. This technique can be used only for a limited number of frequencies or for small model's sizes. The modal approach is more attractive in all other cases.

To take into account absorbent materials, the procedure must rely on complex eigenmode extraction. A five step procedure to perform complex eigenmode extraction is summarized below.

<u>Step 1 : Extraction of undamped modes ($C = 0$)</u>

This operation relies on :

$$(K - \omega_{\mu k}^2 M)\; p_{\mu k} = 0 \qquad k = 1, \; \ldots, \; m \tag{18}$$

where $\omega_{\mu k}$ and $p_{\mu k}$ are respectively (undamped) eigenvalues and eigenvectors. Only first $m$ vectors are selected (with $m$ less than $n$, the number of effective degrees of freedom).

<u>Step 2 : Projection of original eigenproblem into modal space</u>

Let us define the modal base by :

$$P_\mu = (p_{\mu 1}, \; p_{\mu 2}, \; \ldots, \; p_{\mu m}) \tag{19}$$

and express complex eigenmodes with reference to $P_\mu$ :

$$P = P_\mu X \tag{20}$$

The original eigenproblem :

$$(K + i\omega C - \omega^2 M)\, p = 0 \tag{21}$$

can be projected into modal space so that :

$$(\hat{K} + i\omega\hat{C} - \omega^2\hat{M})\, X = 0 \tag{22}$$

where $\hat{K} = P_\mu^T K P_\mu$

$\hat{M} = P_\mu^T N P_\mu$

$\hat{C} = P_\mu^T C P_\mu$

It must be pointed that $\hat{K}$ and $\hat{M}$ matrices are diagonal and the resulting eigenvalue problem is of order $m$ (instead of $n$)

Step 3 : Linearization of the reduced quadratic eigenvalue problem

Problem (21) can be linearized by adding some identity and appers as :

$$\begin{bmatrix} -i\hat{C} & \hat{M} \\ \hat{M} & 0 \end{bmatrix} \begin{pmatrix} X \\ \omega X \end{pmatrix} = \frac{1}{\omega} \begin{bmatrix} \hat{K} & 0 \\ 0 & \hat{M} \end{bmatrix} \begin{pmatrix} X \\ \omega X \end{pmatrix} \tag{23}$$

This reduced linear eigenvalue problem (order $2m$) can be solved using the QR method after reduction to Hessenberg form [16]. Transfromation (20) allows then to compute original eigenvectors.

## 5. FORMULATION OF A RITZ TECHNIQUE FOR DAMPED ACOUSTIC SYSTEMS

Generation of the Ritz vectors basis for damped systems will rely on some equivalent first order formulation of original equations in the time domain

$$M\,\ddot{p} + C\,\dot{p} + K\,p = f \tag{24}$$

By adding the identity

$$M\,\dot{p} - M\,\dot{p} = 0 \tag{25}$$

the second order differential equation can be transformed into a first order one by doubling the size of the system as

$$\begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{pmatrix} \dot{p} \\ \ddot{p} \end{pmatrix} - \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{pmatrix} p \\ \dot{p} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{26}$$

or

$$A \,.\, \dot{z} - B \,.\, z = y \tag{27}$$

with

$$A = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}, \; B = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \tag{28}$$

$$z = \begin{pmatrix} p \\ \dot{p} \end{pmatrix}, \; y = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{29}$$

The usual Ritz vector generation process proposed by Wilson [8] for structural dynamic problems has to be applied to system (20) and is summarized below.

<u>First Vector</u>

Computation of the first vector requires to solve the "static" problem

$$-B \,.\, z_1^{*} = y \tag{30}$$

Solution $z_1^{*}$ has to be normalized versus A matrix so that

$$z_1^{T} A \, z_1 = 1 \tag{31}$$

<u>Additional Vectors</u>

Vector $i$ will result from solution of

$$B \, z_i^{*} = A \, z_{i-1} \tag{32}$$

Solution $z_i^{*}$ has to be A–orthogonalized versus previous vectors. This process requires to compute

$$c_j = z_j^{T} A \, z_i^{*} \qquad (j = 1, \ldots, i-1) \tag{33}$$

and

$$z_i^{**} = z_i^{*} - \sum_{j=1} c_j \, z_j \tag{34}$$

Solution $z_i^{**}$ is scaled so that

$$z_i^{T} A \, z_i = 1 \tag{35}$$

The following remarks have to be done about the generation process :

<u>Remark 1</u>

Solution of system of equations involved in (23) may be formulated as follows

$$\begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \begin{pmatrix} p_1^{*} \\ \dot{p}_1^{*} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{36}$$

and leads to

$$K\, p_i^* = f \tag{37}$$

and

$$M\, \dot{p}_i^* = 0 \tag{38}$$

or

$$\dot{p}_i^* = 0 \tag{39}$$

provided $M$ is not singular. So the solution of original second order system requires only to solve the system (30) of order n. The same facility exists for system (25) rewritten as

$$\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{pmatrix} p_i^* \\ \dot{p}_i^* \end{pmatrix} = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{pmatrix} p_{i-1} \\ \dot{p}_{i-1} \end{pmatrix} \tag{40}$$

and leads to

$$K\, p_i^* = -C\, p_{i-1} - M\, \dot{p}_{i-1} \tag{41}$$

$$M\, \dot{p}_i^* = M\, p_{i-1} \tag{42}$$

or

$$\dot{p}_i^* = p_{i-1} \tag{43}$$

The decomposition of K matrix has to be performed only once. Each Ritz vector determination requires to solve for a new right–hand side. It must be pointed that system (34) involves a real coefficient matrix while right hand side is usually complex.

Remark 2

The orthogonalization procedure requires to compute $c_j$ coefficients using (26). This expression may be reformulated as

$$\begin{aligned} c_j &= (p_j^T,\ \dot{p}_j^T) \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{pmatrix} p_i^* \\ \dot{p}_i^* \end{pmatrix} \\ &= p_j^T\, C\, p_i^* + p_j^T\, M\, \dot{p}_i^* + \dot{p}_j^T\, M\, p_i^* \end{aligned} \tag{44}$$

so, that archival of $C\, p_j$, $M\, p_j$, and $M\, \dot{p}_j$ after generating $j^{th}$ vector appears useful.

Remark 3

The orthogonalization procedure involved in (26, 27) is performed versus the A matrix and allows to diagonalize the projected A matrix but not the B matrix.

Remark 4

Special attention has to be devoted to the treatment of singular systems (with zero frequency modes). This topic and details about computer implementation have to be found elsewhere [14].

Remark 5

The physical meaning related to generation of Ritz vectors is obvious as pointed by Wilson [8]. First vector corresponds to the "static" response while additional vectors take into account inertial and damping effects.

## 6. EVALUATION OF FORCED RESPONSE IN THE FREQUENCY DOMAIN

In the frequency domain ( assumed time dependence like *exp(iwt)* equation (17) can be reformulated as

$$(K - \omega^2 M + i\omega C)\, p = f \tag{45}$$

where $p$ and $f$ denote now pressure and load amplitude vectors.

Rewriting the Ritz vector basis as

$$P = (p_1,\ p_2,\ \dots,\ p_m) \tag{46}$$

where $m$ is the number of vectors selected, solution $p$ of (38) is approximated as

$$p = Px \tag{47}$$

where $x$ is the vector of participation factors. Substitution of (40) into (38) leads after premultiplication by $P^T$ to

$$(\hat{K} - \omega^2 \hat{M} + i\omega \hat{C})\, x = \hat{f} \tag{48}$$

where

$$\begin{aligned} \hat{K} &= P^T K P \\ \hat{M} &= P^T M P \\ \hat{C} &= P^T C P \\ \hat{f} &= P^T f \end{aligned} \tag{49}$$

Solution $x$ of this reduced order system can be sought in some frequency range. Pressure $p$ can then be recovered using (40). As it can be seen, the Ritz vectors generation process, but also the projection process, can be speeded up by memorization of products $K p_i$, $M p_i$ and $C p_i$.

## 7. APPLICATIONS

Muffler System

The finite element method has been widely applied to the study of muffler systems [9 – 12]. The geometry of a simple muffler is given at Figure 1. The following material characteristics are selected : sound speed = 340 m/s, density = 1.225 kg/m$^3$.

Mixed boundary conditions are assumed. In the input section, a unit axial velocity is constrained while the impedance at the output section is simply set to be 416.5 rayls. An axisymmetric model is used for this computation. The two meshes selected for this application (mesh A : 349 nodes, 288 elements; mesh B : 1273 nodes, 1152 elements) are represented at Figure 2. Response is evaluated in the frequency range 10 to 2000 Hz. The insertion loss factor (*IL*) is computed from relation

$$IL = 10 \log \left| \frac{v_i^2}{v_0^2} \right| \tag{50}$$

where $v_i$ and $v_0$ are axial velocities in input and output section respectively. Response's evaluation has been performed using two Ritz vectors bases (10 and 20 vectors). Plot of *IL* factor versus frequency is provided in figure 3 together with the "exact" numerical solution (resulting from direct response option) and the approximate "plane wave" solution.

Computation times for generation of basis and evaluation of forced response are given in Table 1 for the two meshes.

| number of vectors | basis generation | | forced response (200 frequencies) | |
|---|---|---|---|---|
| | mesh A | mesh B | mesh A | mesh B |
| 10 | 15.87 | 77.28 | 11.26 | 12.17 |
| 20 | 49.39 | 231.51 | 29.23 | 29.76 |

Table 1. Computation times (sec, VAX station 3100) for muffler's response using Ritz vectors technique.

The direct response option (solution of whole system at each discrete frequency) gives computation times listed in Table 2.

| case | direct response |
|---|---|
| mesh A | 300.68 |
| mesh B | 2861.65 |

Table 2. Computation times (sec, VAX station 3100) for muffler's response using direct response technique.

Car Compartment

This 2–D application is related to the design of car compartments. Excitation is due to front vibrating panel (normal velocity amplitude = 0.001 m/s). Absorption material (specific admittance = 0.05 + 0.10 i) is used for the seats, carpeting and headliner (as indicated in Figure 4). All remaining boundary surfaces are assumed to be rigid. The Finite element mesh is given at Figure 5. The acoustic response at driver's ear was computed in the 1 – 200 Hz frequency range. Results are presented in Figure 6 for various computational strategies :

- G case 1 : direct response without absorption material,
- G case 2 : direct response with absorption material,
- G case 3 : superposition of 12 Ritz vectors,
- G case 4 : superposition of 14 Ritz vectors,
- G case 5 : superposition of 16 Ritz vectors.

Computation times are given in Table 3.

| number of vectors | basis generation | forced response (200 frequencies) |
|---|---|---|
| 12 | 18.76 | 12.53 |
| 14 | 24.03 | 15.35 |
| 16 | 29.94 | 19.21 |
| direct response (200 frequencies) | | 323.47 |

Table 3. Computation times (sec, VAX station 3100) for car compartment.

## 8. CONCLUSIONS

A refined Ritz vectors technique has been presented for acoustic finite element models. This procedure enables generation of non-modal complex bases. Such bases are less expansive to generate than modal bases and allow to perform efficiently forced response of dissipative systems in the frequency domain.

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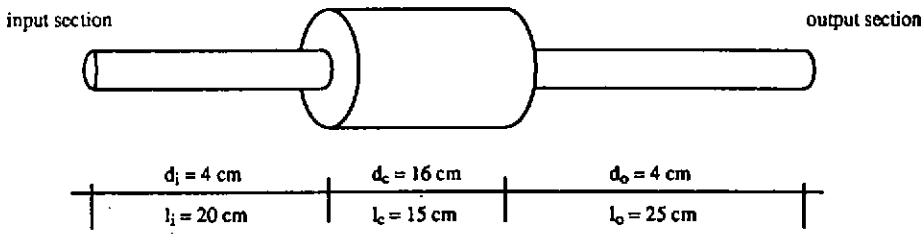


Figure 1. Geometry of a simple muffler

Figure 2. Muffler system – Mesh A Muffler system – Mesh B

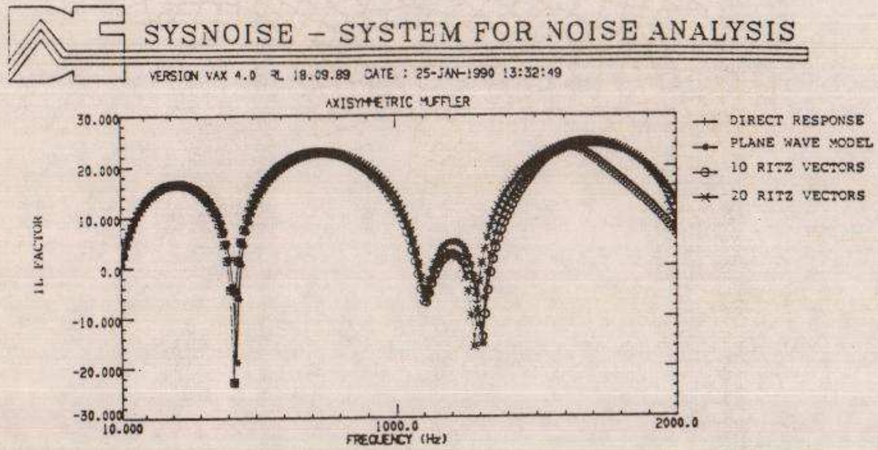


Figure 3. Muffler system – Insertion loss factor

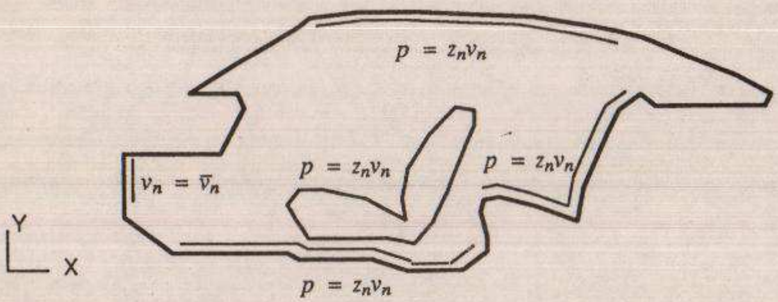


Figure 4. Car compartiment – Boundary conditions

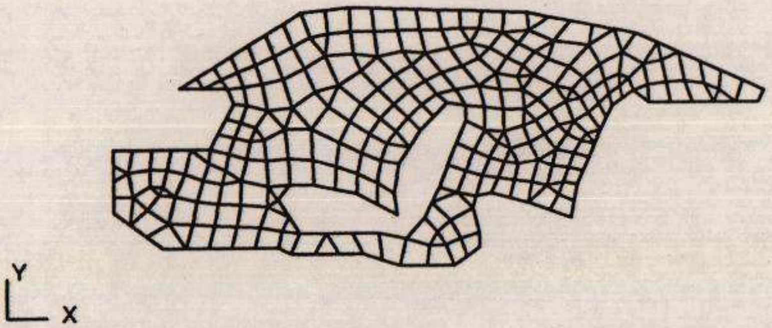


Figure 5. Car compartment – 2–D finite element mesh

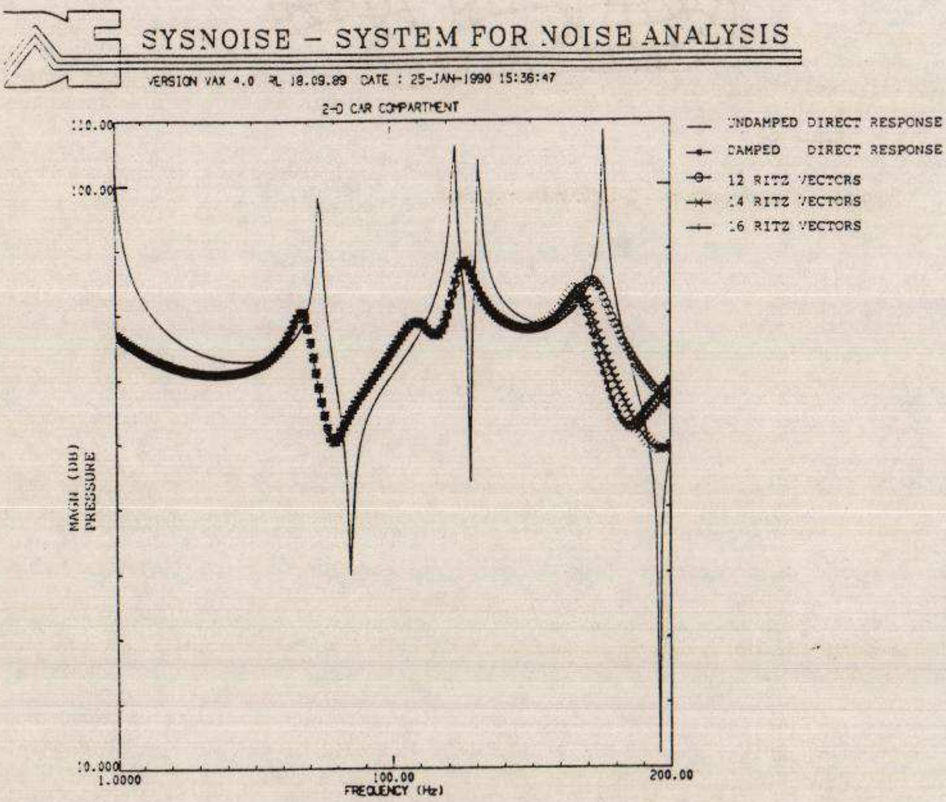


Figure 6. Car Compartment – Acoustic response at driver's ear (pressure in dB, reference value 1 $10^{-5}$ Pa)