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AN EXTENDED BOUNDARY ELEMENT METHOD FOR MODELING SUBMERGED STRUCTURES WITH THIN APPENDAGES

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1. INTRODUCTION

The direct boundary integral formulation [1] is the usual mathematical basis selected for developing a boundary element (BE) model suitable for modeling acoustic radiation from submerged structures. This BE model can also be used with a conventional structural finite element model for modeling fluid-structure interaction effects where it is known that the external (or internal) fluid greatly modifies the structural response (added mass effects).

The extension presented in this paper deals with submerged structures with thin appendages. In such cases, the direct pressure formulation is not well suited and is prone to errors along the thin components. A more rational approach involving pressures and jumps of pressure is presented together with a variational solution scheme. This choice ensures the symmetry of the related fluid matrices and allows the effective coupling with a finite element structural model for handling elasto-acoustic problems. The basic theoretical background is outlined.

2. DIRECT/VARIATIONAL BOUNDARY ELEMENT METHOD

2.1 Direct boundary integral representation

The so-called *direct* boundary integral formulation has been described elsewhere [1]. It allows one to relate the pressure at a field point X (in volume V) to the pressure and the normal pressure gradient on the boundary surface S :

$$p(X) = \int_S \left\{ p(Y)\frac{\partial G(X,Y)}{\partial n_Y} - \frac{\partial p(Y)}{\partial n_Y} G(X,Y) \right\} dS(Y) \tag{1}$$

where G is the fundamental solution of the Helmholtz equation with a point source.

The normal derivative of (1) at some external point X is the starting point for the development of a variational principle and can be formulated as

$$\frac{\partial p(X)}{\partial n_X} = \int_S \left\{ p(Y)\frac{\partial^2 G(X,Y)}{\partial n_X \partial n_Y} - \frac{\partial p(Y)}{\partial n_Y}\frac{\partial G(X,Y)}{\partial n_X} \right\} dS(Y) \tag{2}$$

Using Stokes theorem [2,3,4], (2) can be rewritten as

$$\frac{\partial p(X)}{\partial n_X} = -\int_S \left\{ \frac{\partial p(Y)}{\partial n_Y} (n_X \cdot \nabla_X G(X,Y)) \right\} dS(Y) + k^2 n_X \cdot \int_S \{ n_Y p(Y) G(X,Y) \} dS(Y) + (n_X \wedge \nabla_X) \cdot \int_S \{ (n_Y \wedge \nabla_Y p(Y)) G(X,Y) \} dS(Y) \quad (3)$$

where $\wedge$ denotes the vector product operator.

If the field point X reaches the boundary surface S, one obtains the refined expression

$$k^2 n_X \cdot \int_S \{ n_Y p(Y) G(X,Y) \} dS(Y) + (n_X \wedge \nabla_X) \cdot \int_S \{ (n_Y \wedge \nabla_Y p(Y)) G(X,Y) \} dS(Y) = \frac{\partial p(X)}{\partial n_X} + CPV \int_S \left\{ \frac{\partial p(Y)}{\partial n_Y} (n_X \cdot \nabla_X G(X,Y)) \right\} dS(Y) \quad (4)$$

where the last integral has to be evaluated in the sense of a Cauchy principal value [5] :

$$CPV \int_S \left\{ \frac{\partial p(Y)}{\partial n_Y} (n_X \cdot \nabla_X G(X,Y)) \right\} dS(Y) = \lim_{\varepsilon \to 0} \int_{S-E_\varepsilon} \left\{ \frac{\partial p(Y)}{\partial n_Y} (n_X \cdot \nabla_X G(X,Y)) \right\} dS(Y) \quad (5)$$

In this expression, E_ε is the neighborhood of X on the boundary surface S.

2.2 Variational principle

The variational principle is obtained by multiplying each term of (4) by a virtual increment dp(X) and integrating on the boundary surface S. Using integration by parts leads to the following result [6] :

$$k^2 \int_S \int_S \{ \delta p(X) p(Y) (n_X \cdot n_Y) G(X,Y) \} dS(Y) dS(X) - \int_S \int_S \{ (n_X \wedge \nabla_X \delta p(X)) \cdot (n_Y \wedge \nabla_Y \delta p(Y)) G(X,Y) \} dS(Y) dS(X) = \int_S \delta p(X) \frac{\partial p(X)}{\partial n_X} dS(X) + \int_S \int_S \left\{ \delta p(X) \frac{\partial p(Y)}{\partial n_Y} (n_X \cdot \nabla_X G(X,Y)) \right\} dS(Y) dS(X) \quad (6)$$

or

$$\delta J(p) = 0 \tag{7}$$

where the functional J is given by

$$\begin{aligned} J(p) = {} & \frac{k^2}{2}\int_S\int_S \left\{p(X)p(Y)(n_X \cdot n_Y)G(X,Y)\right\} dS(Y)dS(X) \\ & -\frac{1}{2}\int_S\int_S \left\{(n_X \wedge \nabla_X p(X))\cdot(n_Y \wedge \nabla_Y p(Y))G(X,Y)\right\} dS(Y)dS(X) \\ & -\int_S p(X)\frac{\partial p(X)}{\partial n_X}dS(X) - \int_S\int_S \left\{p(X)\frac{\partial p(Y)}{\partial n_Y}(n_X \cdot \nabla_X G(X,Y))\right\} dS(Y)dS(X) \end{aligned} \tag{8}$$

2.3 Discrete boundary element model

Discretization of functional (8) is based on some approximation of the boundary surface S and the selection of appropriate interpolation (shape) functions for the boundary pressure and the normal velocity (or pressure gradient) :

$$\begin{aligned} & S = \hat{S} = \sum_e S_e \\ & p(X) = N_p(X)P \\ & v_n(X) = N_v(X)V_n \end{aligned} \tag{9}$$

where P and V_n are vectors of nodal pressures and normal velocities, respectively while N_p and N_v are the matrices of interpolation functions.

Note that the normal velocity v_n is related to the normal gradient of pressure through the relation

$$v_n(X) = \frac{1}{-i\rho\omega}\frac{\partial p(X)}{\partial n_X} \tag{10}$$

Substitution of (9-10) into (8) allows one to write the discrete functional $\tilde{J}$ as

$$\tilde{J}(P) = \frac{1}{2}P^T D(k)P + i\rho\omega V_n^T(C + B(k))P \tag{11}$$

where

$$D(k) = k^2 \int_{\hat{S}}\int_{\hat{S}} \left\{ (n_X \cdot n_Y) N_p^T(X) N_p(Y) G(X,Y) \right\} dS(Y) dS(X) - \int_{\hat{S}}\int_{\hat{S}} \left\{ (n_X \wedge \nabla_X N_p(X)) \cdot (n_Y \wedge \nabla_Y N_p(Y)) G(X,Y) \right\} dS(Y) dS(X) \quad (12)$$

$$C = \int_{\hat{S}} \left\{ N_v^T(X) N_p(X) \right\} dS(X) \quad (13)$$

$$B(k) = \int_{\hat{S}}\int_{\hat{S}} \left\{ N_v^T(X) N_p(Y) (n_Y \cdot \nabla_Y G(X,Y)) \right\} dS(Y) dS(X) \quad (14)$$

Stationarity of (11) with respect to P leads to a system of equations :

$$D(k)P = -i\rho\omega \left(C^T + B(k)^T \right) V_n \quad (15)$$

where D(k) and B(k) are frequency dependent matrices. D is symmetric while B is unsymmetric.

3. REFINEMENT OF THE DIRECT FORMULATION FOR STRUCTURES WITH THIN APPENDAGES

3.1 Updated variational statement

If the vibrating structure has thin appendages, the (closed) boundary surface can be divided into several parts related to the main body and the upper and lower faces of the thin components. The methodology to be followed in this case can be formulated with reference to the boundary surface presented at Figure 1.

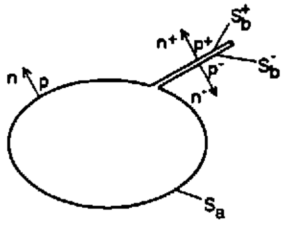

Figure 1 : Structure with a thin appendage

The whole boundary surface S is made from three parts S_a, S_b^+ and S_b^- (as indicated in Figure 1) :

$$S = S_a \cup S_b^+ \cup S_b^- \tag{16}$$

The boundary element model presented in the previous section remains applicable for this case but further simplifications can be introduced by a careful examination of the terms involved in the functional (8).

Looking at the first double surface integral involved in (8), the boundary surface partition (16) allows the related (I_1) term to be decomposed as :

$$\begin{aligned} I_1 = & \int_{S_a}\int_{S_a} F_1(X,Y)\,dS(X)dS(Y) + \int_{S_a}\int_{S_b^+} F_1(X,Y)\,dS(X)dS(Y) + \int_{S_a}\int_{S_b^-} F_1(X,Y)\,dS(X)dS(Y) \\ & + \int_{S_b^+}\int_{S_a} F_1(X,Y)\,dS(X)dS(Y) + \int_{S_b^+}\int_{S_b^+} F_1(X,Y)\,dS(X)dS(Y) + \int_{S_b^+}\int_{S_b^-} F_1(X,Y)\,dS(X)dS(Y) \\ & + \int_{S_b^-}\int_{S_a} F_1(X,Y)\,dS(X)dS(Y) + \int_{S_b^-}\int_{S_b^+} F_1(X,Y)\,dS(X)dS(Y) + \int_{S_b^-}\int_{S_b^-} F_1(X,Y)\,dS(X)dS(Y) \end{aligned} \tag{17}$$

where $F_1(X,Y)$ is given by the expression

$$F_1(X,Y) = \frac{k^2}{2}(n_X \cdot n_Y)p(X)p(Y)G(X,Y) \tag{18}$$

This expression can be further simplified because $S_b^- = S_b^+$ and $n^+ = - n^-$ along the mean surface S_b. The result is that expression (17) can be rewritten as

$$\begin{aligned} I_1 = & \int_{S_a}\int_{S_a} F_1(X,Y)dS(X)dS(Y) + 2\int_{S_a}\int_{S_b^+} F_1(X,Y)dS(X)dS(Y) + 2\int_{S_a}\int_{S_b^-} F_1(X,Y)dS(X)dS(Y) \\ & + \int_{S_b^+}\int_{S_b^+} F_1(X,Y)dS(X)dS(Y) + 2\int_{S_b^+}\int_{S_b^-} F_1(X,Y)dS(X)dS(Y) + \int_{S_b^-}\int_{S_b^-} F_1(X,Y)dS(X)dS(Y) \end{aligned} \tag{19}$$

Defining the jump of pressure variable m along S_b^+ as

$$\mu = p^{+} - p^{-} \qquad (20)$$

allows further simplification of (19) and leads to the final form

$$\begin{aligned} I_1 = &\frac{1}{2}\int_{S_a}\int_{S_a} k^2(n_X \cdot n_Y)p(X)p(Y)G(X,Y)dS(X)dS(Y) \\ &+ \int_{S_a}\int_{S_b^{\prime}} k^2(n_X \cdot n_Y)p(X)\mu(Y)G(X,Y)dS(X)dS(Y) \\ &+ \frac{1}{2}\int_{S_b^{\prime}}\int_{S_b^{\prime}} k^2(n_X \cdot n_Y)\mu(X)\mu(Y)G(X,Y)dS(X)dS(Y) \end{aligned} \qquad (21)$$

The second double surface integral involved in (8) can also be transformed following the same process. The resulting integral (I_2) can be expressed as

$$\begin{aligned} I_2 = &-\frac{1}{2}\int_{S_a}\int_{S_a} (n_X \wedge \nabla_X p(X)) \cdot (n_Y \wedge \nabla_Y p(Y))G(X,Y)dS(X)dS(Y) \\ &- \int_{S_a}\int_{S_b^{\prime}} (n_X \wedge \nabla_X p(X)) \cdot (n_Y \wedge \nabla_Y \mu(Y))G(X,Y)dS(X)dS(Y) \\ &- \frac{1}{2}\int_{S_b^{\prime}}\int_{S_b^{\prime}} (n_X \wedge \nabla_X \mu(X)) \cdot (n_Y \wedge \nabla_Y \mu(Y))G(X,Y)dS(X)dS(Y) \end{aligned} \qquad (22)$$

The third integral in (8) also takes the alternative form

$$I_3 = -\int_{S_a} p(X)\frac{\partial p(X)}{\partial n_X}dS(X) - \int_{S_b^{\prime}} \mu(X)\frac{\partial p(X)}{\partial n_X}dS(X) \qquad (23)$$

while the last integral can be transformed according to

$$
\begin{aligned}
I_4 = & -\int_{S_a}\int_{S_a} F_4(X,Y)dS(X)dS(Y) - \int_{S_b^+}\int_{S_a} F_4(X,Y)dS(X)dS(Y) \\
& - \int_{S_b^-}\int_{S_a} F_4(X,Y)dS(X)dS(Y) - \int_{S}\int_{S_b^+} F_4(X,Y)dS(X)dS(Y) \\
& - \int_{S}\int_{S_b^-} F_4(X,Y)dS(X)dS(Y)
\end{aligned}
\tag{24}
$$

where F_4 stands for the following function

$$
F_4(X,Y) = p(X)\frac{\partial p(Y)}{\partial n_Y}(n_X \cdot \nabla_X G(X,Y))
\tag{25}
$$

Again the identities $S_b^- = S_b^+$ and $n^+ = -n^-$ lead to the final form

$$
\begin{aligned}
I_4 = & -\int_{S_a}\int_{S_a} p(X)\frac{\partial p(Y)}{\partial n_Y}(n_X \cdot \nabla_X G(X,Y))dS(X)dS(Y) \\
& - \int_{S_b^+}\int_{S_a} \mu(X)\frac{\partial p(Y)}{\partial n_Y}(n_X \cdot \nabla_X G(X,Y))dS(X)dS(Y)
\end{aligned}
\tag{26}
$$

The sum of the two last integrals in (25) is zero because normal velocities on S_b^+ and S_b^- are related through

$$
v_n^+ = -v_n^-
\tag{26}
$$

so that the normal pressure gradients along S_b^+ and S_b^- are equal in absolute value but have opposite sign.

3.2 Updated discrete variational form

The discrete form relies on approximating the boundary surface S_a and S_b^+ (Figure 2). Note that S_b^- has been removed from all the integrals involved in the above variational statement and that S_b^+ can be made the same as the mean surface S_b provided the thickness is small versus the acoustic wavelength.

$$S_a = \sum S_a^e \quad S_b = \sum S_b^e \tag{27}$$

Appropriate interpolation functions for the pressure p on S_a and the jump of pressure m on S_b^+ have to be selected :

$$p(X) = N_{pa}(X)P \tag{28}$$

$$\mu(X) = N_{pb}(X)\,J_p \tag{29}$$

where P and J_p are the vectors of nodal pressures and jumps of pressure, respectively.

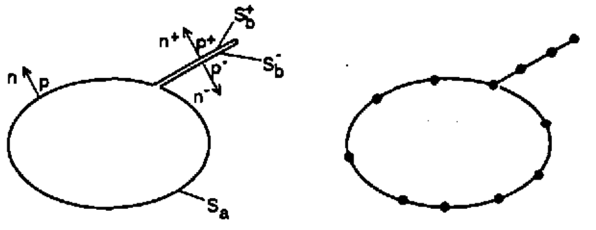

Figure 2 : Boundary element discretization

The discrete updated functional therefore takes the following form

$$\begin{aligned}\tilde{J}(P,J_p) = \frac{1}{2}\left\{P^T D_{aa}(k)P + 2P^T D_{ab}(k)J_p + J_p^T D_{bb}(k)J_p\right\} \\ + i\rho\omega\left\{V_{na}^T C_a P + V_{nb}^T C_b J_p + V_{na}^T B_{aa}(k)P + V_{na}^T B_{ba}(k)J_p\right\}\end{aligned} \tag{30}$$

where

$$\begin{aligned}D_{aa}(k) = k^2 \int_{S_a}\int_{S_a} (n_X \cdot n_Y) N_{pa}^T(X) N_{pa}(Y) G(X,Y)\, dS(X)\, dS(Y) \\ - \int_{S_a}\int_{S_a} \left(n_X \wedge \nabla_X N_{pa}(X)\right) \cdot \left(n_Y \wedge \nabla_Y N_{pa}(Y)\right) G(X,Y)\, dS(X)\, dS(Y)\end{aligned} \tag{31}$$

$$B_{aa}(k) = \int_{\tilde{S}_a}\int_{\tilde{S}_a} N_{va}^T(X) N_{pa}(Y)\left(n_Y \cdot \nabla_Y G(X,Y)\right) dS(X)\, dS(Y) \tag{32}$$

and

$$C_a = \int_{\tilde{S}_a} N_{va}^T(X) N_{pa}(Y)\, dS(X) \tag{33}$$

Similar expressions can be obtained for the other matrices involved in (30).

Stationarity of (30) with respect to P and J_p leads to the following set of equations :

$$\begin{aligned} \frac{\partial \tilde{J}}{\partial P} = 0 &\rightarrow D_{aa}(k)P + D_{ab}(k)J_p = -i\rho\omega C_a^T V_{na} - i\rho\omega B_{aa}^T(k) V_{na} \\ \frac{\partial \tilde{J}}{\partial J_p} = 0 &\rightarrow D_{ab}^T(k)P + D_{bb}(k)J_p = -i\rho\omega C_b^T V_{nb} - i\rho\omega B_{ba}^T(k) V_{na} \end{aligned} \tag{34}$$

or

$$\begin{bmatrix} D_{aa}(k) & D_{ab}(k) \\ D_{ab}^T(k) & D_{bb}(k) \end{bmatrix} \begin{pmatrix} P \\ J_p \end{pmatrix} = -i\rho\omega \begin{bmatrix} C_a^T + B_{aa}^T(k) & 0 \\ B_{ba}^T(k) & C_b^T \end{bmatrix} \begin{pmatrix} V_{na} \\ V_{nb} \end{pmatrix} \tag{35}$$

The solution of this symmetric system of equations (with prescribed boundary normal velocities) gives the nodal pressures on S_a and the jumps of pressure on S_b^*. From these boundary variables, field pressure can be computed using the integral form (1) rewritten as

$$p(X) = \int_{S_a} \left\{ p(Y) \frac{\partial G(X,Y)}{\partial n_Y} - \frac{\partial p(Y)}{\partial n_Y} G(X,Y) \right\} dS(Y) + \int_{S_b^*} \mu(Y) \frac{\partial G(X,Y)}{\partial n_Y} dS(Y) \tag{36}$$

3.3 Limit cases

It is instructive to look at the particular form of (35) when one of the boundary sub-surfaces S_a or S_b^*.vanishes.

If the boundary surface reduces to S_a (S_b^*.= 0), then the system (35) appears as

$$D_{aa}(k)P = -i\rho\omega\left(B_{aa}^T(k) + C_a^T\right)V_{na} \tag{37}$$

while the case $S = S_b^+$ ($S_a = 0$) leads to the system

$$D_{bb}(k)J_p = -i\rho\omega C_b^T V_{nb} \tag{38}$$

already encountered with the so-called *indirect* boundary element formulation [7] which applies to a wide range of thin (open or closed) structures and is available within the SYSNOISE program [8].

3.4 Compatibility requirements

The discrete model for the mixed case relies on the use of appropriate interpolation functions. This requirement is also related to the compatibility of pressure and jump of pressure along the intersection of S_a and S_b^+ sub-surfaces. Looking at the junction line (Figure 3), the following constraint can be formulated

$$\mu_j = p_u - p_l \tag{39}$$

where m_j is the nodal jump of pressure at the junction on $\bar{S}_b^+$ while p_u and p_l are the upper and lower pressures at the closest nodes on $\bar{S}_a$.

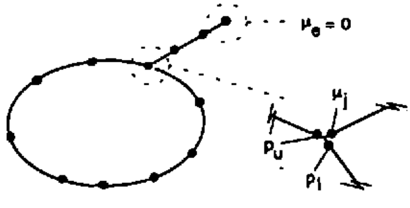

Figure 3 : Junction between S_a and S_b^+ surfaces

Complementary relations like (39) are constraints between nodal variables which can be accounted for using the Lagrange multipliers technique. They have to be supplemented with free edge constraints (m_e=0) at the other end extremities (Figure 3).

4. CONCLUSIONS

An extended acoustic boundary element model has been presented in order to handle radiation from structures with thin appendages. The formulation relies on a direct boundary integral representation. The simplifications related to the thin components have been introduced in the variational statement and leads to a mixed form involving pressure variables on the main body and jump of pressure variables along the mean surface of the thin appendages. These two sets of variables have to be matched along junction lines using appropriate constraints. The resulting boundary element model is characterized by an optimal choice of the boundary variables which contributes, in turn, to reduced model size.

5. REFERENCES

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