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An Elementary Introduction to Finite Element Analysis

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1. Introduction

The finite element method is being increasingly used in the analysis of electro-acoustic transducers for sonar systems, since it is a very powerful technique for situations where the structures are complicated and the materials have anisotropic properties. There are a good number of textbooks covering the more elementary aspects, whereas published papers dealing with the more advanced aspects assume that the reader is familiar with the computational procedures or has access to standard software packages. It is also generally assumed that mainframe computers are essential. This paper shows that fairly simple, albeit special purpose, programs can be written for implementation on a micro computer, and these have been used for analysing various simple transducer structures which are of current interest at Birmingham. The principal limitation is that the complexity of the structures which can be analysed accurately is severely limited due to the restricted size of the available memoyy with fast access. The programs which have been developed permit the analysis of two dimensional and axisymmetric three dimensional structures, comprising several materials which may be anisotropic. Static and dynamic driving forces can be modelled, but not dissipative loading, and electrical loading in any form has been excluded.

2. Principles of the finite element method

The basic principle can be stated very simply as the dividing up of a body into a finite, but large, number of easily defined components, or elements, the behaviour of each of which can be described in discrete terms; in other words a continuous problem is converted into a discrete one. The continuous situation is governed by a system of differential equations; this is equivalent to the division into an infinite number of elements and it requires solution by mathematical manipulation. With the coming of digital computers and their ability to handle large quantities of data at high speeds, discrete methods are now a practical proposition and they are of particular value in situations where the discontinuities within bodies and on their boundaries cannot easily be represented by analytic functions. However, it remains difficult to model a discrete body within an infinite medium such as a radiating or receiving acoustic transducer. For these cases alternative procedures, for example boundary element techniques, may be more appropriate.

The analysis of a continuous problem follows along the line of dividing it into a large number of elements, the behaviour of each of which is specified by a small number of parameters, combining the element parameters together into a global array covering the behaviour of the whole body (a process known as assembly), adding in the external influences and solving the resulting system of simultaneous equations which relate forces and displacements. For a mechanical system the behaviour of each element is defined by the relations between forces and displacements at its nodes, the connecting points of the structure, and the assembly is defined by the requirement for continuity between elements and for local equilibrium at each node. A more complicated system is one in which there is an exchange of energy between electrical and mechanical forms, as in piezoelectric materials, and this would be the proper approach for the investigation of electro-acoustic transducers. However, the simpler technique of considering only the electrically unloaded case with mechanical driving forces has been followed, and this is valid as long as the appropriate (open circuit or constant D) mechanical properties of the piezoelectric material are used.

If the driving functions and responses are sinusoidal, they can be represented by complex functions and so must the terms in the stiffness matrix. If however, there is neither internal damping nor radiation, then the complex terms become real and the method of solution is very little different from that for the static case.

3. Implementation of the finite element method

3.1 General remarks

The basic element used is the constant-strain three-noded triangle applied to two-dimensional bodies and to those with axial symmetry, since this is fairly straight forward to code, and with a sufficiently fine mesh it should give reasonably accurate results. Six-noded triangles could also have been used; for the same number of nodes, but for one quarter of the number of elements, a more accurate representation of the distribution of strains and stresses would have been obtained at the expense of a longer program and an increased amount of data storage. The programs were written in BASIC to run on a BBC microcomputer (model B); they were developed from a FORTRAN program given in ref.2 and modified for the axisymmetric case using the techniques given in ref.1 and for time-varying loads using ideas from ref.3. The first versions were written for isotropic materials, using inputs of Young's modulus and Poisson's ratio, but they were later modified for anisotropic materials using inputs of the stiffness parameters as terms in a 3×3 or 4×4 matrix; this is in a way a more direct method, as in the isotropic case the stiffness parameters still have to be calculated within the program. Furthermore in the data for the piezoelectric ceramics the stiffnesses and compliances are given directly and Poisson's ratio is not quoted. Any number of different materials can be specified.

The disadvantage in using a microcomputer for finite element programs is the limited amount of memory, a large amount of which is required for the storage of input data and intermediate calculations, and to which fast access is essential. With careful setting up of the mesh of elements, it has been found possible to define up to 160 elements with 102 nodes for a simple geometry such as the PZT-4 tube discussed in section 4. These numbers apply to the whole structure being analysed for sensitivity to acceleration, but for sensitivity to pressure it is possible to define a plane of symmetry which undergoes no displacement normal to itself, and hence only half the structure need be analysed with infinite (or at least very high) normal stiffness applied to the plane of symmetry. For the resonances of rectangular bars only one quarter of the full cross-section need be modelled, since only symmetrical modes of vibration can have non-zero electromechanical coupling; the two planes of symmetry are regarded as normally undisplaced boundaries.

3.2 Static and dynamic considerations

The application of static finite element analysis to underwater transducers is limited to hydrophones working at frequencies well below any mechanical resonance, for which the static and dynamic responses are identical, and a simple example is discussed below in section 4.1. The method has its usefulness in analysing complicated hydrophones, the acoustic sensitivity of which cannot be predicted in a simple way. However, most transducers are used in dynamic situations, in which the resonances of the structure are likely to play an important role. If the driving forces vary sinusoidally, then in addition to the internal forces depending on the stiffnesses of the elements there are forces due to their inertia, which is a function of their mass. The element masses are included by adding terms of the form $-\omega^2 M$ to all the diagonal terms in the stiffness matrix, where ω is the angular frequency and M the mass associated with the particular node (this is a proportion of the masses of all the elements connected to that node). The highest frequency that can be used is determined by the fineness of the mesh, because each element must have dimensions which are very small in wavelengths; however, this limitation does not apply to the circumference of an element for purely axisymmetric modes of vibration. Thus non-radiating resonances of a complete structure can be investigated if there are a sufficiently large number of elements. Those for the electrically short-circuited case can be investigated in some cases, an example being sandwich transducers in which the piezoelectric stack is divided into a large number of thin sections wired in parallel so that the short-circuited (constant E) mechanical parameters can be used. By reciprocity these resonant frequencies should be identical with those observed electrically.

3.3 Brief details of the programs

Several independent programs are used, rather than one large multi-purpose one. The main division is between the data preparation and the main programs, which allow for some changes to the data during a run. The main programs are in two groups, for two dimensional and axisymmetric problems respectively, each being subdivided into static and dynamic procedures. There are naturally a substantial number of routines which are common to several programs, but in this way the length of each program is minimised; the choice of the kind of analysis to be done therefore has to be made before the main program is loaded. The format of the data, which is held on tape or disc, is consistent so that the same data can be used in different programs.

4. Results of representative calculations

4.1 The pressure sensitivity of hydrophones

As long as the hydrophone has dimensions which are small in wavelengths and if it is acoustically stiff, then the free-field sensitivity is equal to the pressure sensitivity. Thus the static case can be assumed, and all the complications are avoided which are associated with time-varying fields and the effect of the hydrophone on the incident field, the direction of which would have to be specified. To include all these effects would require a more complicated program and much more data storage. The applied uniform static pressure is simulated in the finite element model by forces at the nodes normal to the local surface, scaled so that the load on each element is proportional to its surface area exposed to the pressure. The solution of the nodal equations, represented by the nodal stiffness matrix and the applied loads, gives the nodal displacements from which the strains and stresses are derived. These are naturally referred to the main axes of the model, i.e. they are calculated as the radial, axial and circumferential components; there are also shear strains in the radial/axial plane, but not in the circumferential plane, since this would not be consistent with the axisymmetry. The output voltage can be obtained simply in this case by multiplying the strains or stresses by the appropriate piezoelectric constants (h or g constants) and by a relevant distance in the direction of poling and summing the contributions from the three axes.

Calculations have been made of the sensitivity of a piezoelectric tube made of PZT-4, 12 mm.OD, 6 mm.ID and 12 mm. long, for the three conditions of freely flooded (for the hydrostatic sensitivity), with the inner surface shielded and the ends exposed, and with the ends closed with rigid caps, for all of which sufficient information is available in the Vernitron catalogue. The tube is silvered on the curved surfaces and poled radially. The proper anisotropic stiffness parameters for the open circuit (constant D) case were used in the stiffness matrix. The mean radial, axial and circumferential strains for a nominal hydrostatic pressure were multiplied by the appropriate piezoelectric strain coefficients, h_{33}, h_{31}, h_{31} respectively, and the results summed and multiplied by the radial thickness to give the generated open-circuit voltage. The hydrostatic sensitivity was calculated directly as the product of the radial thickness and the sum of the piezoelectric stress coefficients ($g_{33} + 2g_{31}$), and the sensitivities for the other conditions were estimated from small scale graphs. The results are tabulated:

	FE method	Maker's data
Hydrostatic case	10.9 μV/Pa	11.7 μV/Pa
Ends exposed	46.3	42
Ends capped	57.2	57

The agreement is seen to be within 10%, which is satisfactory for a simple approach. The figures from the maker's data are independent of the length of the tube; judged from the variation of strain along the length the same is true for the finite element method to an accuracy of better than 5%, i.e. end effects are negligible.

4.2 The acceleration sensitivity of hydrophones

This is an important parameter in some applications and for this axisymmetric model only axial accelerations could be investigated. The sensitivity was calculated in two cases, firstly with the sinusoidal driving force applied at one end and secondly with it applied midway along the inner curved surface. The frequencies used in the analysis were well below any mechanical resonance of the tube, and thus the differential displacements within the tube were small compared with the mean displacement of the complete tube. If the hydrophone is immersed in a liquid, then at frequencies at which its dimensions are very small fractions of a wavelength it may be possible to include the inertia of the surrounding liquid by appropriate additions to the masses associated with the outside nodes, taking into account the direction of the acceleration, but this point has not been investigated. The accuracy of this scheme would rely on the real part of the radiation load being negligibly small compared with the inertial load. No independent theoretical figures were available for comparison, but the mass calculated from the driving force and the acceleration at the point at which it was applied agreed with that calculated from the dimensions with a discrepancy of only 0.04%. For the drive at one end the sensitivity was found to be 1.40 mV/m/sec^2, and for the drive at the centre it was 0.72 μV/m/sec^2, i.e. 66 dB less. Because of the symmetry of the central drive, it would be expected that the sensitivity should ideally be zero, and hence the calculations are consistent with a simple physical model of the situation; they also suggest that the relative round-off errors for the end drive are probably of the order of 0.1%. A very approximate confirmation for the end drive case was obtained in practical measurements on a tube held lightly clamped between nuts and soft washers, for which the sensitivity was found to be 3.1 mV/m/sec^2, a little more than double the theoretical figure.

4.3 Head resonances in sandwich transducers

A brief study has been made of the mechanical resonances in the tapered heads of low frequency, low Q transducers; these are the lowest frequency bending modes. The general shape of a typical head is shown in fig.1a, which also indicates the element mesh used in the analysis; different shapes were generated by multiplying all the coordinates in the thickness direction by a range of constants while keeping the radial dimensions fixed. The ratio of the diameters at the ends is constant at 2.25, and the material is aluminium. The annular surface at the small end, to which the piezoelectric stack would normally be attached, may be either blocked or free, and the axial mechanical drive was round the periphery at the large end. The resonant frequency is that at which the axial displacement at the driven point goes to infinity for a fixed driving force. The results are shown in fig.1b, plotted in normalised form as the diameter/thickness ratio against the ratio of the larger diameter to the wavelength in water at the resonant frequency for a sound velocity of 1500 m/sec. It is interesting to note that the "blocked" and "free" curves intersect; the implication is that for a particular shape, i.e. for the diameter/thickness ratio equal to 6.9, there is least coupling from the driven rim to the smaller end, there being no effect on the resonant frequency whether this surface is blocked or free. Also indicated on the graph are comparable figures for the resonant frequency calculated from material given in an AUWE data sheet.

In the design of a sandwich element it would be sensible to ensure that the parameters associated with the head lie below and to the left of the curves, so that there is the best chance of the head behaving as a rigid piston when it is driven by the piezoelectric stack. This would be more accurately analysed by modelling those parts of a complete transducer which lie in front of the nodal plane, taking as an example a transducer designed by the "lumped element" method.

4.4 Vibrational modes in PZT bars

One problem area in the design of high frequency transducers is in making arrays which have a very wide beamwidth in one plane but a very narrow one in the other. These require elements in the form of long bars with a nearly square cross section. The difficulty arises that there is a significant amplitude of vibration in the width direction comparable with that in the desired thickness direction, and this unwanted radiation is transmitted through the encapsulation and corrupts the directional response. Some theoretical investigation into these modes has been made and the frequency of the lowest resonant mode compared with some practical measurements made some years ago. Only one quarter of the cross section of the bar need be modelled, since for high effective electromechanical coupling the vibration must be symmetrical about both centre lines. The piezoelectric bar is electrically unloaded, the open circuit parameters are used and the bar is infinitely long. The width to thickness ratio was varied, and the results plotted in fig.2 in normalised fashion as width/thickness against width in wavelengths in water; also shown are experimental results. The lowest two modes of vibration are shown, and there is fair agreement between theory and practice, although the theoretical resonant frequencies are lower than the measured ones, whereas they are expected to be higher.

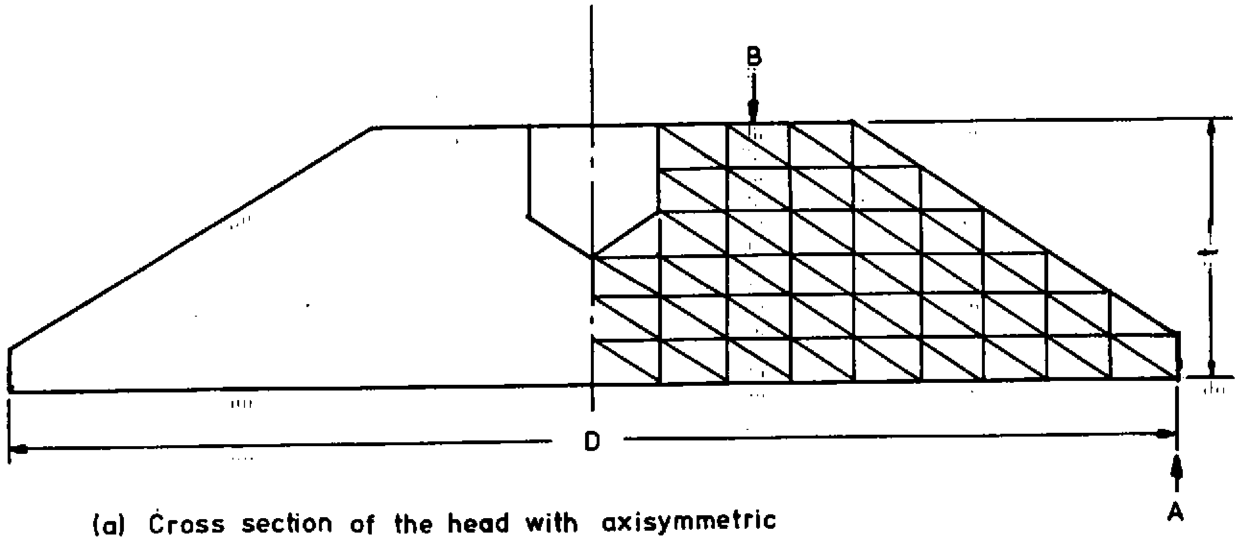
5. References

The following books were consulted in developing the programs:

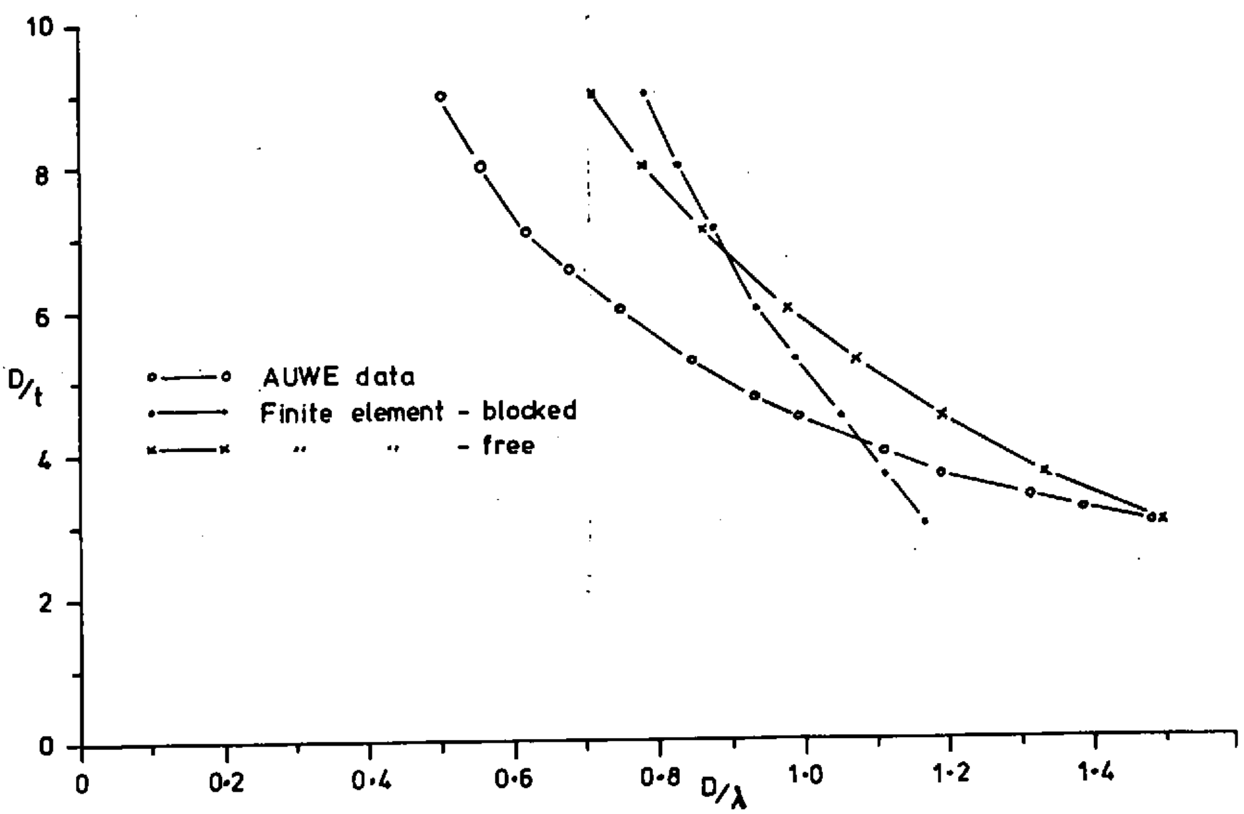
(1) O.C.Zienkiewicz, The finite element method. 3rd edition. McGraw-Hill, Maidenhead, 1977.

(2) Y.K.Cheung & M.F. Yeo, A practical introduction to finite element analysis. Pitman, London, 1979.

(3) B.M. Irons & S. Ahmed, Techniques of finite elements. Ellis Horwood, Chichester, 1980.

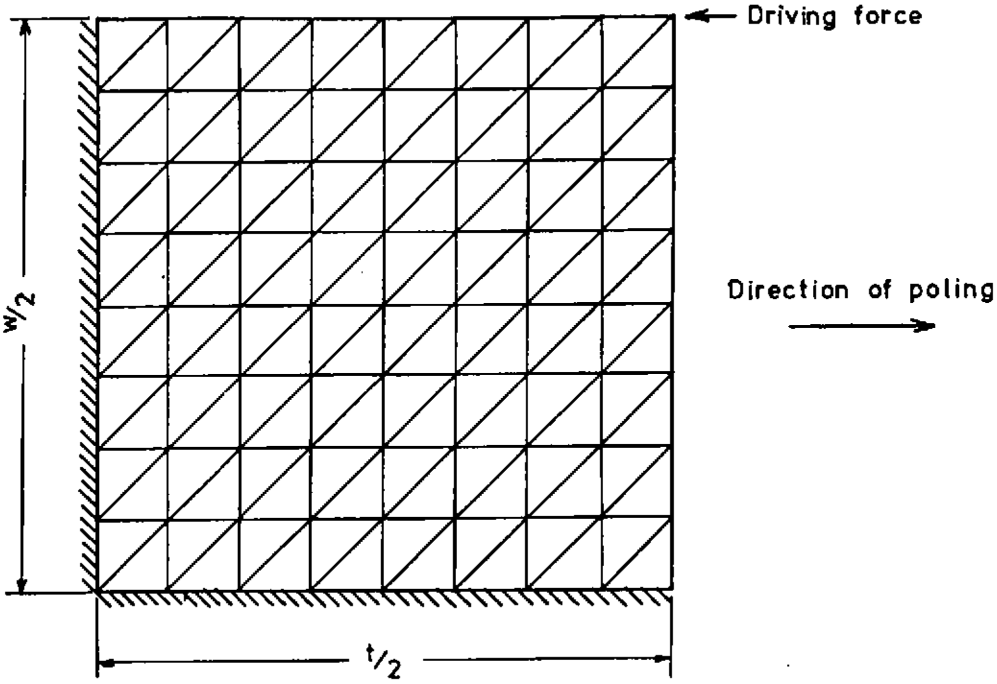


(a) Cross section of the head with axisymmetric finite element mesh
Driving force on circumference at A
Surface B blocked or free

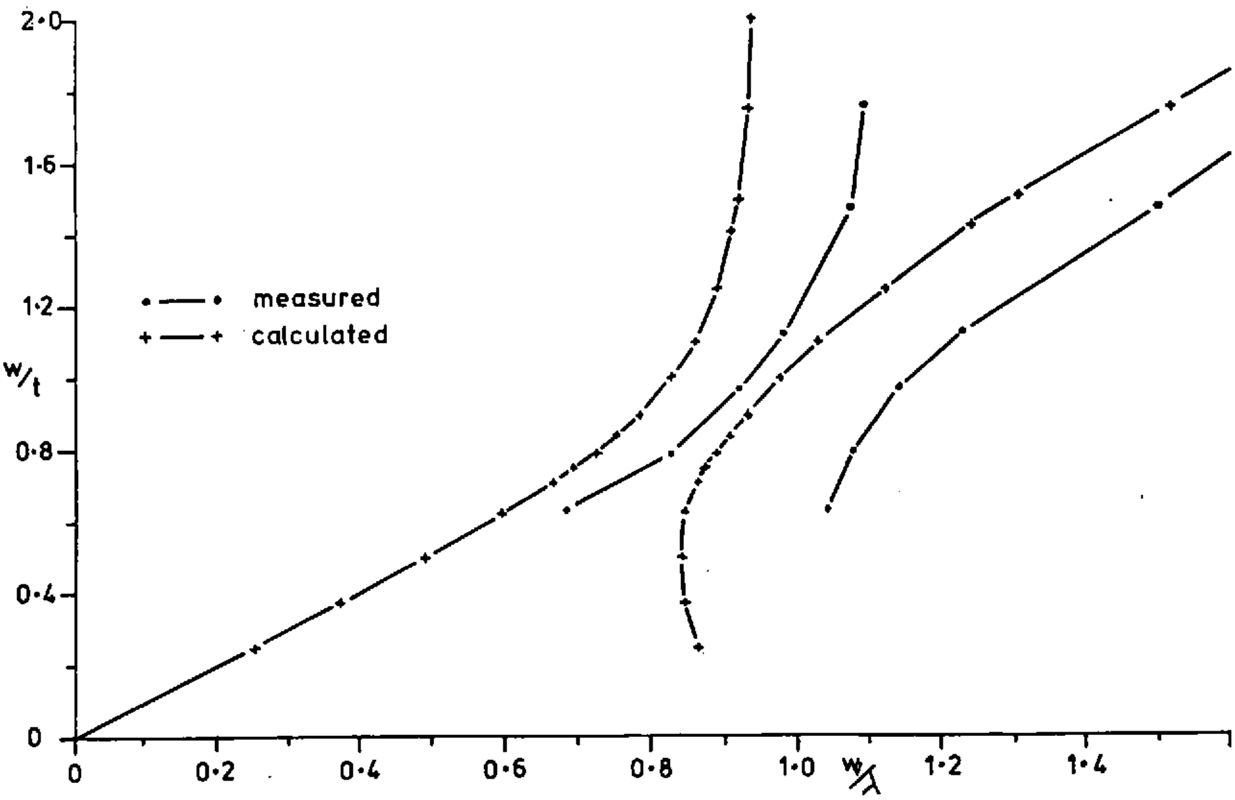


(b) Spectra of computed resonances

Fig. 1 Resonances of a piston transducer head



(a) Cross section of one quadrant of the bar with finite element mesh



(b) Spectra of measured and computed resonances

Fig. 2 Resonances of a long rectangular bar of PZT4