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REFLECTION FROM A VISCOELASTIC LAYER WITH UNDULATING SURFACE

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1. INTRODUCTION

This paper describes a simple method for calculating the amplitude of the reflected wave, when an acoustic plane wave impinges on a viscoelastic layer with a very rough surface, as depicted in Figure 1. The method has previously been applied to the scattering of electromagnetic waves [1].

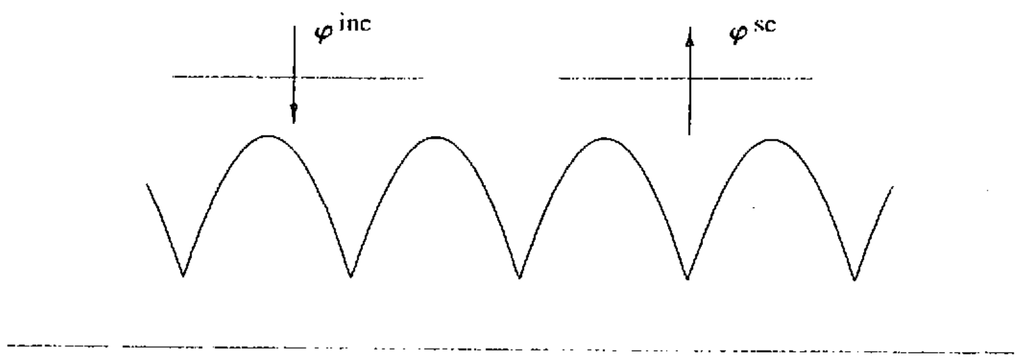

Figure 1. The geometry of the scattering layer.

The method consists of replacing the "interface" region, in which properties vary in the lateral direction between those of the acoustic medium and the solid layer, by a laterally homogeneous but vertically stratified medium, with properties at any particular lateral section chosen as some suitable average of the two local properties at that section. The validity of this approximation can be proved, in the asymptotic limit of a very rough interface. Examples demonstrate, however, that the approximation generates quite accurate results, even for an interface with the degree of roughness illustrated in the figure. No rigorous proof is available – nor can one be expected – but the practical utility of the simple approximate method is verified by comparison with more precise (and much more laborious) calculations, based on boundary integral equations, as well as with the results of experiments.

These ideas are explained in the sections that follow. First, in Section 2, it is shown how the reflection coefficient for reflection from a vertically stratified medium can be found directly from the solution of an ordinary differential equation of Ricatti type. Section 3 explains its application to reflection from the rough layer. Then, Section 4 gives a brief account of the exact formulation, in terms of boundary integral equations. These require formulae for Green's functions which can be expressed in terms of Fourier series. The details are complicated and are relegated to an appendix. Finally, some results are presented in Section 5.

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2. REFLECTION FROM A VERTICALLY STRATIFIED LAYER

Throughout this work, the incident waves are assumed to have sinusoidal time dependence; thus, all physical quantities are interpreted as the real parts of corresponding complex quantities, multiplied by a factor $\exp(-i\omega t)$, which is suppressed. Relative to Cartesian axes $Ox_1x_2x_3$, the layer occupies the region $d < x_3 < h$ and the acoustic medium through which the incident and reflected waves propagate occupies $h < x_3 < \infty$. The acoustic medium (inviscid fluid) is characterised by a bulk modulus κ_f and density ρ_f, both assumed uniform. The layer has the stress-strain relations

$$\sigma_{ij} = c_{ijkl}e_{kl},$$

where strain components e_{ij} are related to displacement components u_i by

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}),$$

and density ρ_l. In these equations, the summation convention applies to repeated suffixes and $_{,j}$ represents $\partial/\partial x_j$. The layer is viscoelastic and the constants c_{ijkl} are complex and depend upon the circular frequency ω, as well as displaying variation with the coordinate x_3; the density ρ_l is real and independent of ω but depends on x_3.

For the present application, the layer will display transversely isotropic symmetry about an axis in the x_1-direction. In such a case, the stress-strain relations can be written in the form

$$\begin{aligned} \tfrac{1}{2}(\sigma_{22} + \sigma_{33}) &= k(e_{22} + e_{33}) + le_{11}, \\ \sigma_{11} &= l(e_{22} + e_{33}) + ne_{11}, \\ \sigma_{22} - \sigma_{33} &= 2m(e_{22} - e_{33}), \quad \sigma_{23} = 2me_{23}, \\ \sigma_{12} &= 2pe_{12}, \quad \sigma_{13} = 2pe_{13}. \end{aligned} \tag{1}$$

An acoustic wave incident normally on the layer then generates a displacement, in the layer and in the fluid, whose only non-zero component is u_3, and this is a function of x_3 only. It is described by the following set of equations.
In the fluid $(x_3 > h)$:

$$\sigma_{33}' + \rho_f\omega^2 u_3 = 0; \quad \sigma_{33} = \kappa_f u_3'. \tag{2}$$

In the layer $(d < x_3 < h)$:

$$\sigma_{33}' + \rho_l\omega^2 u_3 = 0; \quad \sigma_{33} = (k + m)u_3'. \tag{3}$$

Here, a prime signifies differentiation with respect to x_3. In addition, σ_{33} and u_3 are continuous at the interface $x_3 = h$, and some "homogeneous" boundary condition is assumed at $x_3 = d$. In general, this can be expressed as an impedance relation

$$\sigma_{33}(d) = -i\omega Z(d)u_3(d), \tag{4}$$

including the possibilities $Z(d) = 0$ or $Z(d) = \infty$, to allow for zero traction or zero displacement at $x_3 = d$.

A convenient way to formulate this problem is to set

$$\sigma_{33}(x_3) = -i\omega Z(x_3)u_3(x_3). \tag{5}$$

Then, in the layer, it follows that

$$Z' = i\omega\left\{\frac{Z^2}{k+m} - \rho_l\right\}. \tag{6}$$

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The advantage of the impedance formulation is that, if the layer in fact terminated at height x_3, the coefficient R for reflection of the incident wave back into the fluid would be

$$R = \frac{Z_f - Z}{Z_f + Z}, \tag{7}$$

where Z_f is the intrinsic impedance of the fluid, $Z_f = (\rho_f \kappa_f)^{1/2}$. It is still more convenient to employ R, as given by (7), as the unknown in place of Z: by elementary manipulation, R satisfies the equation

$$R' = \frac{-i\omega\rho_l}{2Z_f}\left[\left(\frac{Z_f^2}{Z_l^2} - 1\right)(1 + R^2) - 2R\left(\frac{Z_f^2}{Z_l^2} + 1\right)\right], \tag{8}$$

where $Z_l = [\rho_l(k + m)]^{1/2}$. All that is required to solve the problem is to solve the first-order differential equation (8) (which is of Ricatti type), subject to the given condition at $x_3 = d$. For this purpose, the simple implicit scheme obtained by replacing R' by $(R_{n+1} - R_n)/\Delta$ (where Δ is the step length), R^2 by $R_{n+1}R_n$ and $2R$ by $R_{n+1} + R_n$, has the interesting property that, when the layer is elastic so that Z_l is real, if R_n has modulus 1 for some n, corresponding to perfect reflection, then R_n has modulus 1 exactly, for all n. Explicitly, the recurrence relation is

$$R_{n+1} = \frac{R_n + \frac{i\omega\rho_f\Delta}{2Z_f}\left[\left(\frac{Z_f^2}{Z_l^2} + 1\right)R_n - \left(\frac{Z_f^2}{Z_l^2} - 1\right)\right]}{1 - \frac{i\omega\rho_f\Delta}{2Z_f}\left[\left(\frac{Z_f^2}{Z_l^2} + 1\right) - \left(\frac{Z_f^2}{Z_l^2} - 1\right)R_n\right]}. \tag{9}$$

The formulation given above can be generalised fairly easily to the case of a wave incident at any angle, and to a layer whose constants c_{ijkl} have no particular symmetry. Limitations of space prevent its presentation.

3. APPLICATION TO THE ROUGH LAYER

Suppose that the interface between the layer and the fluid is *very* rough, as depicted in Figure 2.

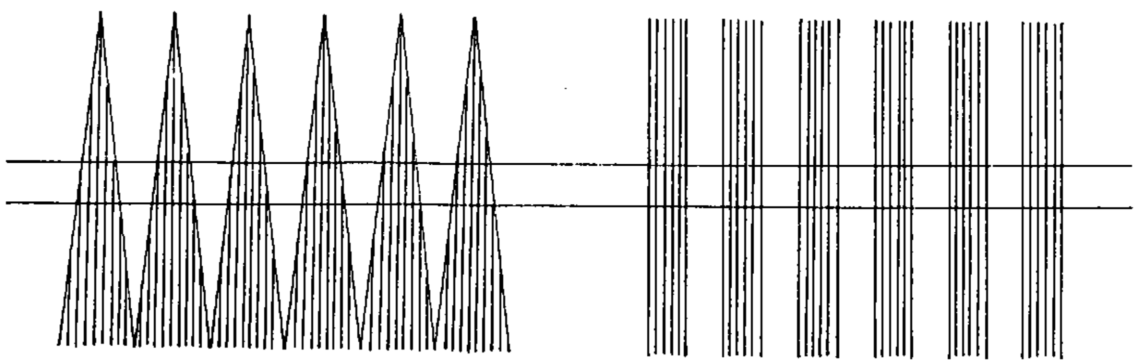

Figure 2. A very rough interface and its local approximation as a laminate.

At any particular section, specified by a value of x_3, it *appears* locally to have the form of a laminated medium, with the volume fraction of the material comprising the layer taking a value $\nu(x_3)$, say. So long as the waves in the layer have wavelength much larger than the separation of adjacent peaks, the region in the vicinity of the section at height x_3 can be approximated as *homogeneous*, with constants appropriate to those of a laminate with volume fraction of layer material $\nu(x_3)$. The formulae are well-known: see, for example, [2]. Locally, the "laminate" must be in equilibrium. The three stress components σ_{11}, σ_{12} and σ_{13}, and the three strain components e_{22}, e_{33} and e_{23} cannot depend on x_1 and so must equal their mean values in the section at height x_3. In detail, in the case that the materials comprising the laminate are isotropic, with Lamé constants λ, μ, rearrangement of the stress-strain relations yields

$$\begin{aligned}
\tfrac{1}{2}(\sigma_{22}+\sigma_{33}) &= \frac{(3\lambda+2\mu)\mu}{\lambda+2\mu}(\langle e_{22}\rangle+\langle e_{33}\rangle)+\frac{\lambda}{\lambda+2\mu}\langle\sigma_{11}\rangle,\\
\sigma_{22}-\sigma_{33} &= 2\mu(\langle e_{22}\rangle-\langle e_{33}\rangle), \quad \sigma_{23}=2\mu\langle e_{23}\rangle,\\
e_{11} &= \frac{1}{\lambda+2\mu}[\langle\sigma_{11}\rangle-\lambda(\langle e_{22}\rangle+\langle e_{33}\rangle)],\\
e_{13} &= \frac{1}{2\mu}\langle\sigma_{13}\rangle, \quad e_{12}=\frac{1}{2\mu}\langle\sigma_{12}\rangle.
\end{aligned} \tag{10}$$

Here, the angled brackets denote the average of the quantity enclosed, over the section at height x_3. Equations (10) can now be so averaged and rearranged to provide relations between the mean values of stress and strain, of the form of equations (1), with

$$\begin{aligned}
k &= \langle\frac{(3\lambda+2\mu)\mu}{\lambda+2\mu}\rangle+\langle\frac{1}{\lambda+2\mu}\rangle^{-1}\langle\frac{\lambda}{\lambda+2\mu}\rangle^{2}, \quad l=\langle\frac{1}{\lambda+2\mu}\rangle^{-1}\langle\frac{\lambda}{\lambda+2\mu}\rangle,\\
n &= \langle\frac{1}{\lambda+2\mu}\rangle^{-1}, \quad m=\langle\mu\rangle, \quad p=\langle\frac{1}{\mu}\rangle^{-1}.
\end{aligned} \tag{11}$$

The effective density of the laminate is obtained by averaging the momentum density; continuity of the velocity induces the simple result that the effective density is just the mean density, $\langle\rho\rangle$.

These formulae apply when *all* components of the laminate are solid, and are not restricted to a two-component laminate. If, however, one of the components is a fluid, for which $\mu=0$, it follows from (11) that $p=0$, so that the laminate cannot support relative shear of its planes of symmetry. The "next term" in the series therefore becomes significant: this would involve the construction of a higher-order theory, allowing for bending stiffness of the solid laminae. For the case of normal incidence, discussed in Section 2, this problem does not arise: the relevant parameter is $k+m$, and this is provided by equations (11). It is perhaps worth noting that, although the solid/fluid laminate presents an interesting theoretical challenge, the troughs in the structure would be likely to be filled in with some solid material, in any practical application.

It may be noted that equations (11) retain validity, regardless of the detailed structure of the laminate. No restriction to periodicity is implied, and the approximate formulation of the reflection problem thus applies to *any* rough surface.

4. EXACT FORMULATION

Although the approximate formulation developed above applies to any sufficiently rough surface, an exact description requires the selection of some particular form. Here, equations are presented for a surface with periodic structure, such as depicted in Figure 1. To be specific, the bottom surface of the layer is taken to be the plane $x_3=d$, while the top surface is defined by the equation $x_3=F(x_1)$, where F is a periodic function with period L.

In the case of normal incidence, all of the fields will have period L and so can be represented as Fourier series. In the case of the fluid, the acoustic wave can be described in terms of a velocity potential ϕ, which may be split into incident and scattered components:

$$\phi=\phi^{\rm inc}+\phi^{\rm sc}. \tag{12}$$

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The incident wave has the form

$$\phi^{\rm inc} = A\exp(-{\rm i}k_f x_3), \tag{13}$$

where $k_f = \omega(\rho_f/\kappa_f)^{1/2}$. The scattered wave varies also with x_1, so

$$\phi^{\rm sc} = \sum_{n=-\infty}^{\infty} a_n \exp\{-[2\pi{\rm i}nx_1 + (4\pi^2n^2 - k_f^2L^2)^{1/2}x_3]/L\}, \tag{14}$$

the square root being chosen so that it is either positive real or negative imaginary, to correspond either to evanescent or to upward-travelling waves. The form (14) ensures that $\phi^{\rm sc}$ satisfies the reduced wave equation

$$\nabla^2\phi^{\rm sc} + k_f^2\phi^{\rm sc} = 0. \tag{15}$$

There is only one upward-travelling wave (with $n = 0$), so long as $k_f L < 2\pi$; in this case, it is legitimate to discuss a single reflection coefficient, $R = a_0/A$. The simple theory outlined above can be expected to apply *at most* to this frequency range – and it is optimistic to expect it to provide a good representation except for some low-frequency part of this.

The field $\phi^{\rm sc}$ can be expressed at a general point, in terms of the pressure and velocity at the interface $x_3 = F(x_1)$, by introducing a Green's function G_f, which is periodic with period L with respect to x_1 and satisfies

$$\nabla^2 G_f + k_f^2 G_f = \delta(x_1 - x_1')\delta(x_3 - x_3'); \quad -L/2 < x_1, x_1' < L/2. \tag{16}$$

Elementary calculation, based on the divergence theorem, yields

$$\phi^{\rm sc}(x_1', x_3') = \int_C [G_f\nabla\phi.\mathbf{n} - \phi\nabla G_f.\mathbf{n}]{\rm d}s, \tag{17}$$

where C represents the curve $\{x_3 = F(x_1); -L/2 < x_1 < L/2\}$, $\mathbf{n}$ is its unit normal, with components $(-F'(x_1), 1)/(1 + F'^2)^{1/2}$ and s is arc length along the curve. The argument follows exactly that given in [1] for the case of electromagnetic waves. It should be noted that the particle velocity in the fluid is $\mathbf{v} = \nabla\phi$, and the pressure is $p = {\rm i}\omega\rho_f\phi$.

A similar integral representation can be developed for the displacement $\mathbf{u}$ in the layer. This satisfies the equations

$$(\lambda + \mu)u_{j,ji} + \mu u_{i,jj} + \rho_l\omega^2 u_i = 0, \tag{18}$$

and the corresponding Green's tensor (which is periodic with period L with respect to x_1) satisfies

$$(\lambda + \mu)G_{jp,ji} + \mu G_{ip,jj} + \rho_l\omega^2 G_{ip} + \delta_{ip}\delta(x_1 - x_1')\delta(x_3 - x_3') = 0; \quad -L/2 < x_1, x_1' < L/2. \tag{19}$$

The representation is

$$u_p(x_1', x_3') = \int_C [G_{ip}\sigma_{ij}n_j - u_i\Sigma_{ijp}n_j]{\rm d}s, \tag{20}$$

where σ_{ij} are the stresses associated with u_i:

$$\sigma_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i})$$

and Σ_{ijp} are similarly associated with G_{ip}. It is important to note that G_{ip} must also satisfy the relevant homogeneous boundary condition at the bottom of the layer, $x_3 = d$.

The continuity conditions at the interface $x_1 = F(x_3)$ are:

$$\begin{aligned} -{\rm i}\omega\mathbf{u}.\mathbf{n} &= \nabla\phi.\mathbf{n}, \\ \sigma_{ij}n_in_j &= -p, \\ \sigma_{ij}l_in_j &= 0, \end{aligned} \tag{21}$$

where $\mathbf{l}$ denotes the unit tangent vector with components $(1, F'(x_1))/(1+F'^2)^{1/2}$.

It is convenient to define

$$u = \mathbf{u.n} \quad \text{and} \quad v = \mathbf{u.l},$$

so that $u_i = un_i + vl_i$. The representations (18) and (20) may now be expressed in terms of the interface quantities p, u and v:

$$\phi^{sc}(x_1', x_3') = \int_C [\mathrm{i}\omega u G_j + \frac{\mathrm{i}}{\omega\rho_f} p\nabla G_f.\mathbf{n}]\mathrm{d}s, \tag{22}$$

$$u_p(x_1', x_3') = \int_C [-pn_iG_{ip} - un_i\Sigma_{ijp}n_j - vl_i\Sigma_{ijp}n_j]\mathrm{d}s. \tag{23}$$

Integral equations for p, u and v now follow by letting (x_1', x_3') tend to C 'from above' in (22) and 'from below' in (23). Since

$$p = \mathrm{i}\omega\rho_f[\phi^{sc} + A\exp(-\mathrm{i}k_f x_3)],$$

equation (22) implies

$$p(x_1', x_3') - \mathrm{i}\omega\rho_f A\exp(-\mathrm{i}k_f x_3) = \int_C [-p\nabla G_f.\mathbf{n} + \omega^2\rho_f u G_f]\mathrm{d}s. \tag{24}$$

Similarly, (23) yields

$$u(x_1', x_3') = -n_p' \int_C [pn_iG_{ip} + un_in_j\Sigma_{ijp} + vl_in_j\Sigma_{ijp}]\mathrm{d}s \tag{25}$$

and

$$v(x_1', x_3') = -l_p' \int_C [pn_iG_{ip} + un_in_j\Sigma_{ijp} + vl_in_j\Sigma_{ijp}]\mathrm{d}s, \tag{26}$$

where $\mathbf{l}'$ and $\mathbf{n}'$ are evaluated at $(x_1', F(x_1'))$.

There is no great originality in the formulation, or solution, of these integral equations. Similar equations have been developed in [3], for example. The greatest difficulty relates to the computation of the Green's functions; here, the present account perhaps displays some slight novelty. The expressions from which the Greens functions were computed are given in the Appendix.

5. RESULTS

It has been remarked above that the simple approximate formulation of Sections 2 and 3 can be expected to work well, when the period of the surface undulations is sufficiently small. Thus it is required that $k_f L << 1$ but there is no restriction on wavelength relative to layer thickness, $k_f(h-d)$. Figures 3, 4 and 5 display results that substantiate this: the surface undulations are as shown in the insets. They violate the asymptotic assumption illustrated in Figure 2, and yet the comparison between the approximation and the more exact solution, obtained by solving numerically the integral equations, demonstrates the practical utility of the simple approximation. Figure 3 shows results for a layer on a rigid foundation, so that the displacements are zero when $x_3 = d$. The mean thickness of the layer is 1cm, the period is 1cm and the vertical distance between peaks and troughs is 1cm. The constants λ and μ were taken to be

$$\lambda = 186.667 - 69.0933\mathrm{i}, \quad \mu = 20.0 - 82.6\mathrm{i}(\mathrm{MPa}),$$

independent of frequency, and $\rho_l = 1800$ Kgm^{-3}. The fluid was taken as water ($\kappa_f = 2250$ MPa, $\rho_f = 1000$ Kgm^{-3}). Figure 4 shows a similar comparison. The parameters are the same but the lower surface of the layer is traction-free. The curves show echo reduction (in dB) against frequency (in KHz). Figure 5 gives the same comparison, when the layer is backed by fluid. The values of λ, μ and ρ_l were taken as above but the geometry of the layer is as shown in the inset; the dimensions correspond to an experimental setup reported in [4].

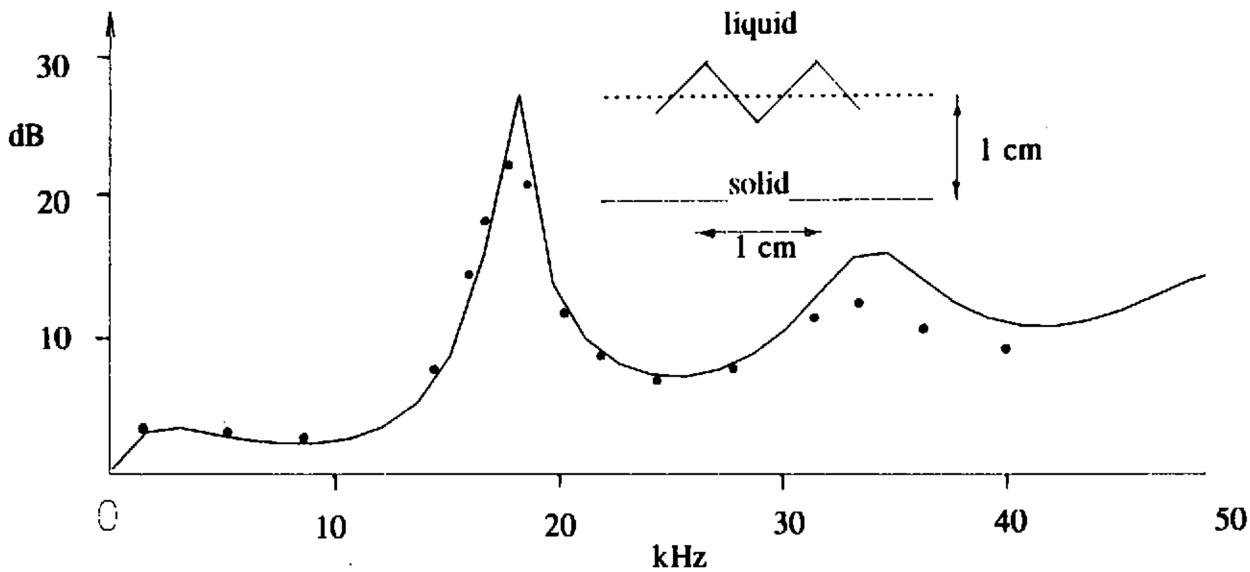

Figure 3. Comparison of simple approximation (solid curve) with solution of the integral equation (discrete points) for a layer with the geometry illustrated, in the case of rigid backing.

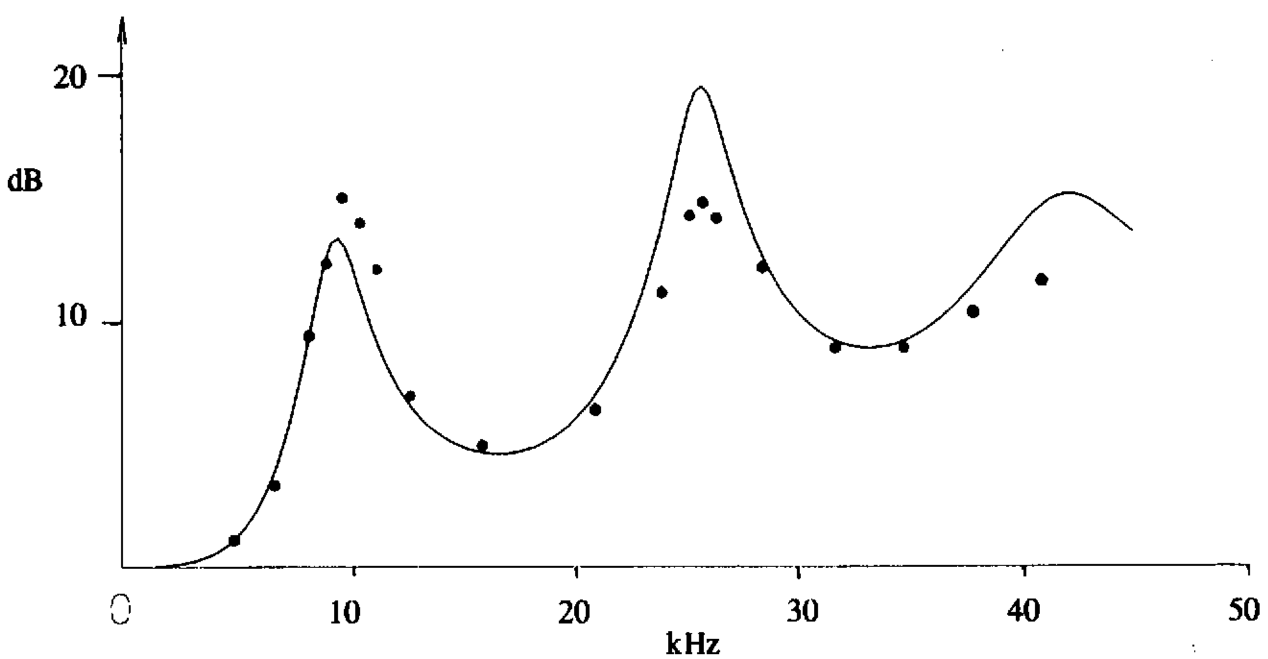

Figure 4. As for Figure 3, except that the back of the layer is traction-free.

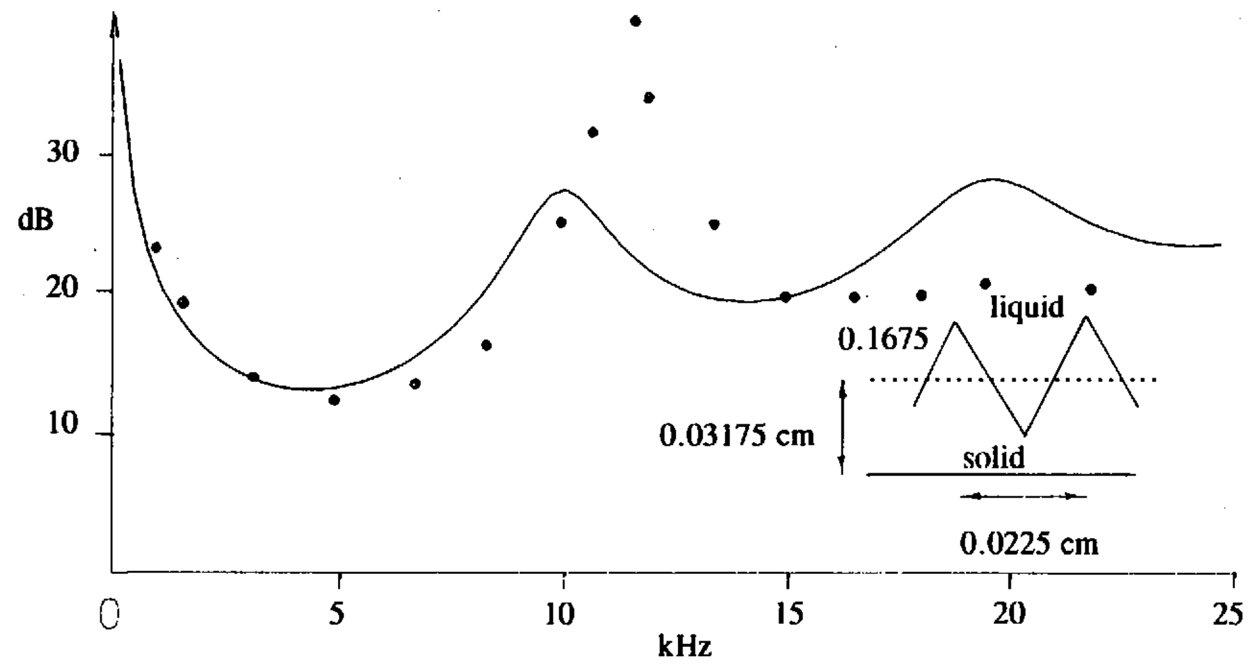

Figure 5. As for Figures 3 and 4, except that the dimensions of the layer are as illustrated, and the layer is backed with fluid.

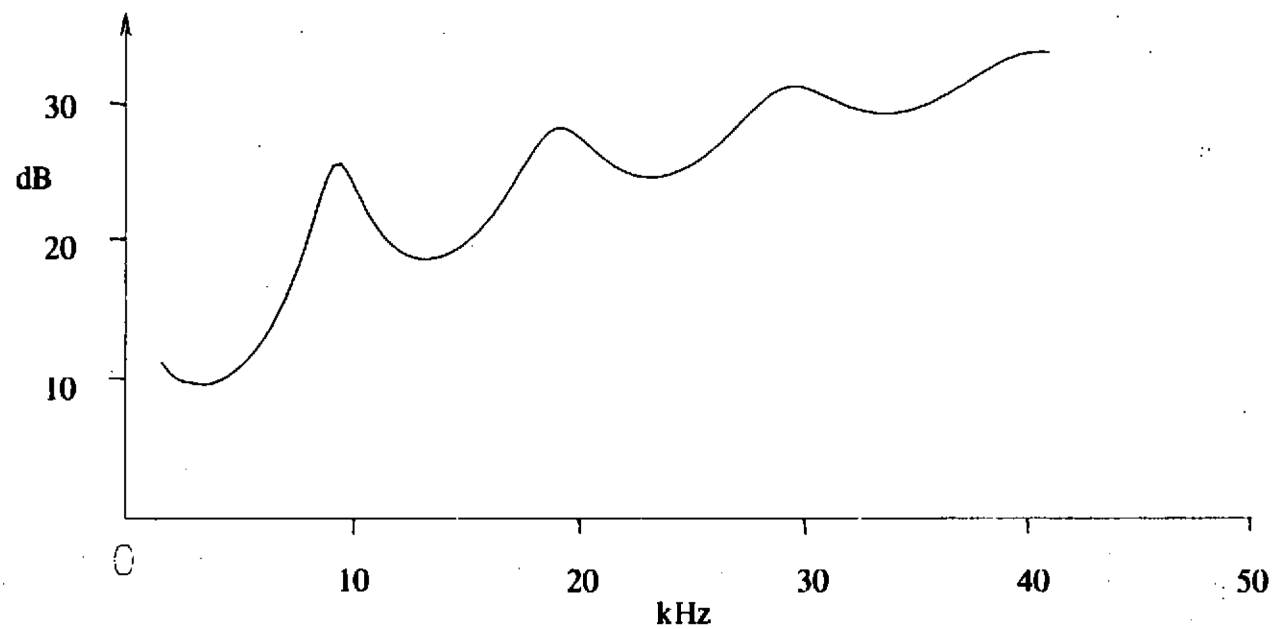

Figure 6. Prediction from the simple approximation, with physical parameters chosen to match experimental data.

The integral equations become increasingly difficult to solve as frequency increases, due to the need for a discretization fine enough to resolve not only the detail of the interface but also the variation of the field within a wavelength. This dictated the range of frequencies over which the comparison was made. Figure 6 shows, finally, a prediction, over a somewhat greater frequency range, made just from the simple approximation. The geometry is exactly as for Figure 5, but this time, frequency-dependent values were incorporated for λ and μ, to match an experiment reported in [4]. The figure bears a striking resemblance (considering the experimental variation in the results for different orientations of the layer in the water tank) to Figure [6] in reference [4].

6. APPENDIX: GREEN'S FUNCTIONS

6.1 The acoustic Green's function

Since G_f is periodic and so is associated with a periodic array of sources, it can be given in the form $G_f(x_1 - x_1', x_3 - x_3')$, where $G_f(x_1, x_3)$ satisfies

$$\nabla^2 G_f + k_f^2 G_f = \delta(x_1)\delta(x_3); \quad -L/2 < x_1 < L/2.$$

Furthermore, it has a Fourier series representation

$$G_f(x_1, x_3) = \sum_{n=-\infty}^{\infty} g_n(x_3) \exp(-2\pi i n x_1/L), \tag{A1}$$

where $g_n(x_3)$ satisfies the ordinary differential equation

$$g_n'' + (k_f^2 - 4\pi^2 n^2/L^2) g_n = \frac{1}{L}\delta(x_3).$$

Hence,

$$g_n(x_3) = -\frac{\exp[-(4\pi^2 n^2 - k_f^2 L^2)^{1/2}|x_3|/L]}{2(4\pi^2 n^2 - k_f^2 L^2)^{1/2}}. \tag{A2}$$

The resulting explicit series for G_f is, unfortunately, slowly convergent when $x_3 = 0$. This reflects the fact that G_f has a logarithmic singularity at each of the points $(nL, 0)$, which is the same as that in the corresponding static Green's function G_s, which satisfies

$$\nabla^2 G_s = \delta(x_1)\delta(x_3). \tag{A3}$$

Equation $(A3)$ has a solution in closed form:

$$G_s(x_1, x_3) = \frac{1}{2\pi} \mathrm{Re}\left\{ \ln\left[\sin\left(\frac{\pi z}{l}\right)\right]\right\}, \tag{A4}$$

where $z = x_1 + i x_3$.

A Fourier series for G_s may also be constructed. It can be found from that for G_f by taking $k_f = 0$, except for the term with $n = 0$, which can be found by considering the asymptotic form of $(A4)$ as $x_3 \to \infty$. Finally, by adding and subtracting the two representations for G_s, equation $(A1)$ can be given in the explicit form

$$G_f(x_1, x_3) = \frac{1}{2\pi}\mathrm{Re}\left\{ \ln\left[\sin\left(\sin\frac{\pi z}{L}\right)\right]\right\} - \frac{|x_3|}{2L} + \frac{1}{2\pi}\ln 2 + \frac{\exp(i k_f x_3)}{2 i k_f L}$$
$$+ \sum_{n=1}^{\infty} \left\{ \frac{\exp[-(2\pi n|x_3|/L)]}{2\pi n} - \frac{\exp[-(4\pi^2 n^2 - k_f^2 L^2)^{1/2}|x_3|/L]}{(4\pi^2 n^2 - k_f^2 L^2)^{1/2}} \right\} \cos\left(\frac{2\pi n x_1}{L}\right). \tag{A5}$$

The series in $(A5)$ converges absolutely for all x_3. Corresponding series for the derivatives of G_f with respect to x_1 or x_3 are obtained by differentiating $(A5)$.

6.2 Green's functions for the layer
The construction of G_{ip} follows, in principle, the pattern established above in calculating G_f. The details, however, are more complicated; only the results are recorded here.

6.2.1 The semi-infinite layer. Consider first the case $d \to -\infty$, for which the boundary condition is replaced by the requirement that G_{ip} contains only down-going waves as $x_3 \to -\infty$. The series expression, analogous to $(A1)$, is

$$G_{ip}(x_1, x_3) = \frac{1}{L} \sum_{n=-\infty}^{\infty} g_{ip}(2\pi n/L, x_3) \exp(-2\pi i n x_1/L), \qquad (A6)$$

where

$$\begin{aligned}
g_{11}(\xi, x_3) &= \frac{i}{2\mu} \frac{\exp[i(\omega^2/\beta^2 - \xi^2)^{1/2}|x_3|]}{(\omega^2/\beta^2 - \xi^2)^{1/2}} - \frac{i\xi^2}{2\rho_l\omega^2}\Big[(\omega^2/\beta^2 - \xi^2)^{-1/2} \exp[i(\omega^2/\beta^2 - \xi^2)^{1/2}|x_3|] \\
&\qquad - (\omega^2/\alpha^2 - \xi^2)^{-1/2} \exp[i(\omega^2/\alpha^2 - \xi^2)^{1/2}|x_3|]\Big], \\
g_{13}(\xi, x_3) = g_{31}(\xi, x_3) &= \frac{i\xi}{2\rho_l\omega^2}\Big[\exp[i(\omega^2/\beta^2 - \xi^2)^{1/2}|x_3|] - \exp[i(\omega^2/\alpha^2 - \xi^2)^{1/2}|x_3|]\Big]\operatorname{sgn}(x_3), \\
g_{33}(\xi, x_3) &= \frac{i}{2\mu} \frac{\exp[i(\omega^2/\beta^2 - \xi^2)^{1/2}|x_3|]}{(\omega^2/\beta^2 - \xi^2)^{1/2}} - \frac{i}{2\rho_l\omega^2}\Big[(\omega^2/\beta^2 - \xi^2)^{1/2} \exp[i(\omega^2/\beta^2 - \xi^2)^{1/2}|x_3|] \\
&\qquad - (\omega^2/\alpha^2 - \xi^2)^{1/2} \exp[i(\omega^2/\alpha^2 - \xi^2)^{1/2}|x_3|]\Big],
\end{aligned} \qquad (A7)$$

with

$$\alpha^2 = (\lambda + 2\mu)/\rho_l, \quad \beta^2 = \mu/\rho_l. \qquad (A8)$$

The series $(A7)$ converges slowly when $x_3 = 0$. Again, it is appropriate to add and subtract representations for the static Green's function, G^s_{ip}. In closed form,

$$\begin{aligned}
2\mu G^s_{11} &= -\frac{(\lambda + 3\mu)}{2\pi(\lambda + 2\mu)} \mathrm{Re}\left\{\ln\left[\sin\left(\frac{\pi z}{L}\right)\right]\right\} + \frac{x_3(\lambda + \mu)}{2L(\lambda + 2\mu)} \mathrm{Im}\left\{\cot\left(\frac{\pi z}{L}\right)\right\}, \\
2\mu G^s_{13} &= \frac{x_3(\lambda + \mu)}{2L(\lambda + 2\mu)} \mathrm{Re}\left\{\cot\left(\frac{\pi z}{L}\right)\right\}, \\
2\mu G^s_{33} &= -\frac{(\lambda + 3\mu)}{2\pi(\lambda + 2\mu)} \mathrm{Re}\left\{\ln\left[\sin\left(\frac{\pi z}{L}\right)\right]\right\} - \frac{x_3(\lambda + \mu)}{2L(\lambda + 2\mu)} \mathrm{Im}\left\{\cot\left(\frac{\pi z}{L}\right)\right\}.
\end{aligned} \qquad (A9)$$

The coefficients in the corresponding series are given by

$$[g^s_{ip}(\xi, x_3)] = \frac{\exp(-|\xi||x_3|)}{4\pi(\lambda + 2\mu)} \begin{bmatrix} \frac{(\lambda+3\mu)}{|\xi|} - (\lambda + \mu)|x_3| & \frac{i(\lambda+\mu)\xi x_3}{|\xi|} \\ \frac{i(\lambda+\mu)\xi x_3}{|\xi|} & \frac{(\lambda+3\mu)}{|\xi|} + (\lambda + \mu)|x_3| \end{bmatrix} \qquad (A10)$$

if $\xi \neq 0$, and

$$[g^s_{ip}(0, x_3)] = \begin{bmatrix} -\frac{|x_3|}{2\mu} & 0 \\ 0 & -\frac{|x_3|}{2(\lambda+2\mu)} \end{bmatrix} + \frac{(\lambda + 3\mu)(\ln 2)L}{4\pi\mu(\lambda + 2\mu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \qquad (A11)$$

(Here, the matrices have entries with suffixes (11), (13), (31) and (33)). The required representation for G_{ip} is now

$$G_{ip}(x_1,x_3) = G^{s}_{ip}(x_1,x_3) + \frac{1}{L}\sum_{n=-\infty}^{\infty} \exp(-2\pi i n x_1/L)\left\{g_{ip}\left(\frac{2\pi n}{L},x_3\right) - g^{s}_{ip}\left(\frac{2\pi n}{L},x_3\right)\right\}. \quad (A12)$$

6.2.2 Layer of finite thickness. Green's function for a finite layer is obtained from Green's function for a semi-infinite layer by adding to it an 'image' field, which satisfies equations (19) without the delta-function source and contains only up-going or evanescent waves as $x_3 \to +\infty$, such that the total field satisfies the prescribed boundary conditions at $x_3 = d$. Translation invariance with respect to x_1 implies that Green's function for a point force at (x_1', x_3') takes the form $G_{ip}(x_1 - x_1', x_3, x_3')$; it is possible, therfore, to take $x_1' = 0$ without loss. Then, Green's function can be given in the form

$$G_{ip}(x_1,x_3,x_3') = \frac{1}{L}\sum_{n=-\infty}^{\infty} \hat{g}_{ip}(2\pi n/L, x_3, x_3')\exp(-2\pi n x_1/L), \quad (A13)$$

where

$$\hat{g}_{ip}(\xi,x_3,x_3') = g_{ip}(\xi,x_3-x_3') + g^{\mathrm{im}}_{ip}(\xi,x_3,x_3') \quad (A14)$$

and g_{ip} are given by ($A7$). Satisfaction of the homogeneous equations corresponding to (19), and the condition for up-going waves, is achieved by taking

$$\begin{aligned} g^{\mathrm{im}}_{1p}(\xi,x_3,x_3') &= -\xi A_p \exp[\mathrm{i}k_\alpha(x_3-d)] + k_\beta B_p \exp[\mathrm{i}k_\beta(x_3-d)], \\ g_{3p}(\xi,x_3,x_3') &= k_\alpha A_p \exp[\mathrm{i}k_\alpha(x_3-d)] + \xi B_p \exp[\mathrm{i}k_\beta(x_3-d)], \end{aligned} \quad (A15)$$

where

$$k_\alpha = (\omega^2/\alpha^2 - \xi^2)^{1/2} \quad \text{and} \quad k_\beta = (\omega^2/\beta^2 - \xi^2)^{1/2}. \quad (A16)$$

The constants A_p, B_p depend on the boundary conditions.

For zero displacements at $x_3 = d$:

$$\begin{aligned} A_p &= \frac{1}{k_\alpha k_\beta + \xi^2}\left[\xi g_{1p}(\xi,d-x_3') - k_\beta g_{3p}(\xi,d-x_3')\right], \\ B_p &= \frac{-1}{k_\alpha k_\beta + \xi^2}\left[k_\alpha g_{1p}(\xi,d-x_3') + \xi g_{3p}(\xi,d-x_3')\right]. \end{aligned} \quad (A17)$$

For zero tractions at $x_3 = d$:

$$\begin{bmatrix} A_p \\ B_p \end{bmatrix} = \frac{\mathrm{i}}{\Delta}\begin{bmatrix} 2\mu\xi k_\beta & k_\beta^2 - \xi^2 \\ -[\lambda\xi^2 + (\lambda+2\mu)k_\alpha^2] & 2\xi k_\alpha \end{bmatrix} \times \begin{bmatrix} -(g_{1p,3} - \mathrm{i}\xi g_{3p}) \\ -\mathrm{i}\lambda\xi g_{1p} + (\lambda+2\mu)g_{3p,3} \end{bmatrix}, \quad (A18)$$

where

$$\Delta = 4\xi^2\mu k_\alpha k_\beta + (k_\beta^2 - \xi^2)[\lambda\xi^2 + (\lambda+2\mu)k_\alpha^2]. \quad (A19)$$

If the layer is backed by fluid, the constants A_p, B_p have to be chosen so that the correct continuity conditions are satisfied at $x_3 = d$: these are that the shear traction is zero, while normal traction and velocity are continuous with those in the fluid occupying $x_3 < d$, into which a downgoing wave of the general form $(A1)$ is transmitted. Completion of the algebra leads to

$$\begin{bmatrix} A_p \\ B_p \end{bmatrix} = \frac{\mathrm{i}}{\Delta_1} \begin{bmatrix} 2\mu\xi k_\beta + \omega^2\xi\rho_f/k_f & k_\beta^2 - \xi^2 \\ -[\lambda\xi^2 + (\lambda+2\mu)k_\alpha^2 + \omega^2\rho_f k_\alpha/k_f] & 2\xi k_\alpha \end{bmatrix} \times$$
$$\times \begin{bmatrix} -(g_{1p,3} - \mathrm{i}\xi g_{3p}) \\ -\mathrm{i}\lambda\xi g_{1p} + (\lambda+2\mu)g_{3p,3} + \mathrm{i}(\omega^2\rho_f/k_f)g_{3p} \end{bmatrix}, \qquad (A20)$$

where

$$\Delta_1 = 2\xi^2 k_\alpha(2\mu k_\beta + \omega^2\rho_f/k_f) + (k_\beta^2 - \xi^2)\left[\lambda\xi^2 + (\lambda+2\mu)k_\alpha^2 + \omega^2\rho_f k_\alpha/k_f\right]. \qquad (A21)$$

These expressions reduce to $(A18)$, $(A19)$ when $\rho_f = 0$.

The series for the 'image' part of the Green's function converges absolutely, when it is evaluated at any point of the upper interface $x_3 = F(x_1)$; there is no need to speed its convergence as was done for the 'semi-infinite layer' part, given by $(A12)$.

7. REFERENCES

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