AN INTRODUCTION TO ADAPTIVE ARRAY PROCESSING

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1. Introduction

Array processing systems which can respond to an unknown interference environment are currently of considerable interest. The fundamentals of such systems are by no means new - basically they depend on Weiner's filter theory - but application in practice has been limited both by technology and by the lack of robust algorithms rapid enough for real time operation. Rapid strides in the past decade or so in the twin fields of electronic components and computer technology have changed the situation considerably by offering the possibility of complicated signal processing in real time at economic costs. This has led to an increased interest in adaptive processing and although it is probably still finding its main application in the defence field the civilian applications are growing.

An array comprises a set of sensors, the outputs of which are combined in some way to produce a desired effect, e.g. a set of beams 'looking' in various directions. The sensors may be of many forms, e.g. acoustic transducers for sonar, monopole aerials for h.f. reception, microwave horns in a radar system, the influence of the application on the processing required concerns the technology not the principles involved. Sensors may be distributed in space in various ways but the two most common are the linear array, normally a set of equally spaced sensors in a straight line and the circular array in which the sensors are arranged uniformly in a circular pattern. Although the distribution of sensors obviously affects the problem the effect is only of second order. The linear array, apart from being the most common in practice, is also the simplest to describe and understand and will be used in this paper to introduce the subject. The introduction to the theory will also be limited to 'band pass' systems, i.e. systems in which the signals can be described in terms of a carrier with a complex amplitude. In the theory the carrier will thus not be explicit and the calculations will be based on the complex numbers representing the amplitude at any time.

2. Two Element Array

As shown in Figure 1 the output of a two element array for a plane wave arriving from an angle θ will be

$$y = 2 \cos \phi/2$$

where $\phi = \frac{2\pi d}{\lambda} \sin \theta$

and d = spacing between elements

For example, if $d = \frac{\lambda}{2}$ then $\phi = \pi \sin \theta$ and the polar diagram is as shown in Figure 1.

Thus if a wanted signal was at $\theta = 0^\circ$ (broadside position) and an interference at $\theta = 30^\circ$ the ratio of gain of wanted signal to interference would be $\sqrt{2}$ (3 dB). A strong interference would thus mask a weak signal.

However, if we weight the signals from the two elements as shown in Figure 2 it is possible to fix the gain of the array to the wanted signal as unity while adjusting the weights to minimise the interference. In this simple case, since there is only one interfering signal, it is possible to null it completely, viz:

$$y = w_1 e^{j\phi/2} + w_2 e^{-j\phi/2}$$

Constraint $y = 1$ when $\theta = 0$ i.e. $\phi = 0$

Thus $w_1 + w_2 = 1$

For null at say ϕ_1

$$w_1 e^{j\phi_1/2} + w_2 e^{-j\phi_1/2} = 0$$

Hence $w_1 = \frac{1}{1 - e^{j\phi_1}}$ and $w_2 = \frac{1}{1 - e^{-j\phi_1}}$

e.g. if $d = \lambda/2$ and $\theta = 30^o$

then $\phi_1 = \pi/2$

$$y = \sqrt{2}\cos\{\phi/2 + \pi/4\}$$

as is shown in Figure 2.

3. Multi Element Array

We can extend the principles discussed for the two-element array for many elements but in doing so it is most convenient to resort to matrix algebra. Figure 3 shows the basic system and in Figure 4 we define the various vectors representing the signals in the array. S is the vector representing the wanted signal, W the weights and N the unwanted signals/ interference/noise. It is important to note that when we wish to 'look' in a particular direction the wanted signal is the signal coming from that direction (or nearly so - a factor we will consider later) and signals coming from other directions are regarded as interference. When we choose to "look' in another direction the role of wanted signal and interference will be interchanged. The output from the array is given by

$$y = \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 \quad .... \quad \omega_k x_k$$

$$= W^T X$$

To keep the gain to the wanted signal constant we require

$$W^T C = 1$$

where C is a'steering' vector to steer the beam in the direction of S, e.g. for a linear array will comprise a set of phase changes

i.e. $C = \{e^{-jr\phi}\}$ $\quad r = o \; ..... \; k-1$

Hence we have a set of equations which allow us to minimise the contribution to y of interference sources while maintaining the gain to the wanted signal.

If the number of sources is less than the number of elements then the system can set nulls at the interfering source directions as was done in the two element case for the one interfering source. Otherwise the system will minimise the mean square output of y.

The expected value of y^2 is given by

$$E(y^2) = E\left[W^T X\right] = E\left[W^T X\ X^T W\right]$$

$$= W^T\ E\left[X X^T\right]\ W$$

$$= W^T\ R_{xx}\ W \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where R_{xx} is the correlation matrix.

Using Lagrange Multipliers we can define a cost function

$$H(W) = W^T\ R_{xx}\ W + \lambda\ (W^T C - 1) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

by differentiation we can obtain the optimum value of W to minimise $E(y^2)$

$$W_{opt} = \frac{R_{xx}^{-1}\ C}{C^T\ R_{xx}^{-1}\ C} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Power output $P = E\left[y^2\right]$

$$= W^T\ R_{xx}\ W$$

Hence for optimum condition

$$P = W_{opt}^T\ R_{xx}\ W_{opt}$$

$$R_{xx}\ W_{opt} = \frac{C}{C^T\ R_{xx}^{-1}\ C}$$

$$\therefore \quad P = \frac{W_{opt}^T\ C}{C^T\ R_{xx}^{-1}\ C} = \frac{1}{C^T\ R_{xx}^{-1}\ C} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

4. Recursive Solutions

It can be shown that this solution is analogous to the matched filter approach for detecting a pulse in coloured noise. Thus if we are dealing with a stationary system and are not in a hurry the solution is fairly straightforward. The strategy would be to steer the system in as many directions as required and calculate the optimum 'weight vector' for each direction. This would enable the power received from each direction to be measured under the optimal conditions. Normally in practice neither the condition of stationarity nor of ample time hold and what we require is a system which can adapt to changing conditions and preferably do this very rapidly.

The most obvious approach is to use a recursive method and to update the weights as new data is processed. We have seen that our criterion is to minimise the expected mean square of the output (ℓ.m.s.) subject to the constraint. If we imagine a multidimensional space whose ordinates are the weight vector components then for a given environmental situation, i.e. a particular group of incoming signal interference/noise, a set of contours can be drawn for a given power output. These will form a 'bowl' and we are trying to seek the minimum (bottom) of this bowl subject to the constraint requirement. Figure 5 illustrates this point using a very simple situation of a weight vector comprising two real components. One of the standard methods of searching for the minimum is to move along the direction of steepest descent. We require, however, also to satisfy our constraint and so we approach the solution in a series of double steps. Step 1 alters the weights in such a manner as to move along the direction of steepest descent and then the second step is to correct the weights to obey the constraint criterion. From Equation (1) we obtain the gradient vector

$$V = R_{XX} W \qquad \text{.............................. (5)}$$

We can now write our recursive formulation as

$$\left.\begin{aligned} W'_{k+1} &= W_k - \beta R_{xx} W_k \\ W_{k+1} &= W'_{k+1} + C\{I - C^T W'_{k+1}\}/C^T C \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Again this is illustrated by the simplified situation of Figure 6. The constraint requires the tip of the vector W to lie on a straight line. In the more normal multi-dimensional situation this would be a surface. Several other important factors can be noted from this. In the case of one interfering signal the vector W is adjusted until it is orthogonal to the interference. Thus as the interference approaches closer to the wanted direction the magnitude of the vector must increase considerably to maintain the two conditions of orthogonality and obeying the constraint. This property can be used to prevent nulling of a wanted signal which is not exactly in the direction to which the array is being steered.

Conventional weighting, i.e. a weight vector of equal co-ordinates (usually $\frac{1}{N}C$) is also illustrated on the diagram of Figure 6. If we let

$$W = W_c + \omega$$

where W_c = conventional weight vector

then since $C^T W = 1$

and $C^T W_c = 1$

then $C^T \omega = 0$

Thus the vector W comprises the conventional weight vector W_c together with a component ω which is orthogonal to the constraint vector

The equation for optimal weighting contains the factor R_{xx}^{-1} and yet we will not even know R_{xx}, i.e. $E(XX^T)$. What we will have is an estimate of R_{xx} from the data available so far. One approach is to replace R_{xx} in the steepest descent algorithm by the present value of XX^T.

i.e. $W'_{k+1} = W_k - \beta X_k X_k^T W_k$

but $X_k^T W_k = y_k$

$\therefore$ $W'_{k+1} = W_k - \beta X_k y_k$

This provides the basis of a simple feedback system which is illustrated in Figure 7.

An alternative approach is to use a recursive method to estimate R_{xx} or better still to estimate R_{xx}^{-1}, the latter method avoiding both the processing time and inherent difficulties of inverting a matrix. There are many other methods of tackling this problem of adaption at least one for every worker in the field! However, none is a panacea since the 'best' solution varies according to the environment in which it is to be applied. In simulation it is fairly easy to set up data for which a particular algorithm works well but just as easy to produce data for which it does not!

We have not discussed the effect of truncation and quantization in the processing nor the need for robustness in dealing with, for example, variations of the sensitivities or positions. Suffice it to say that these introduce further complications but are capable of analysis and control.

In general wide-band systems can be dealt with by using an F.F.T. processor to provide a set of narrow-band systems to which the adaptive method is applied individually.

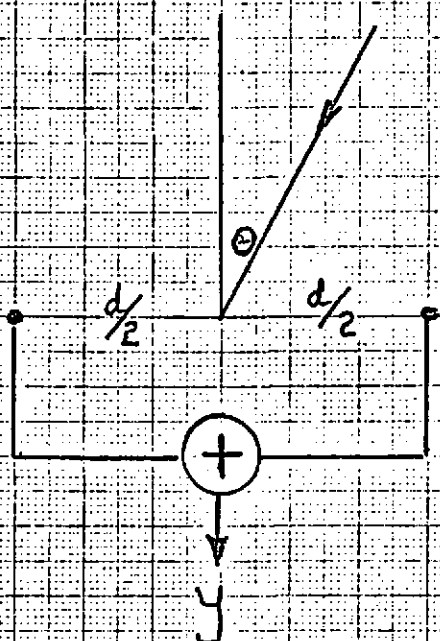
5. References

This introductory paper to the subject of adaptive arrays has been based on a paper presented by the author and his colleague, Dr. J.E. Hudson, at the NATO Study Institute on Signal Processing held at Portovenere, Italy in September 1976. The Proceedings have been published by the Reidel Publishing Co., Holland and interested readers are referred to this for further reading and many references.

6. Acknowledgement

The author would like to acknowledge the many sources from which information has been obtained and in particular the assistance of his colleague, Dr. J.E. Hudson.

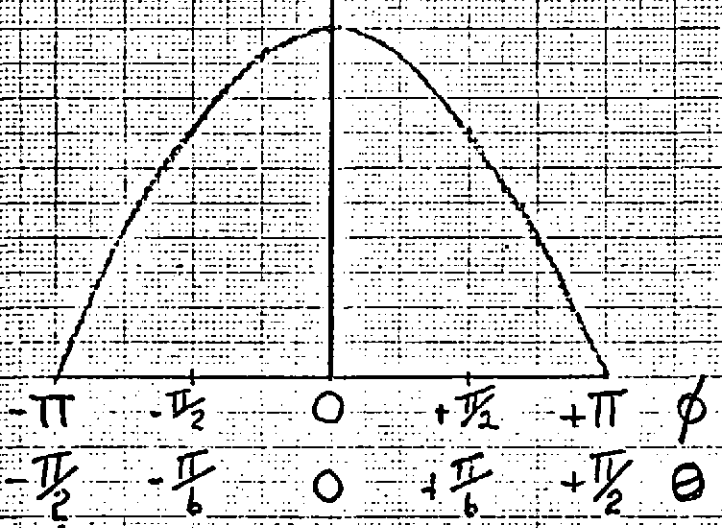
Figure 1 Two element array

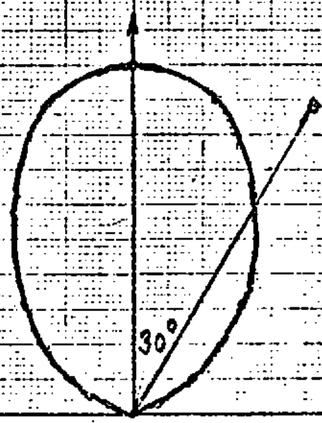


$$y = e^{j\phi/2} + e^{-j\phi/2}$$

$$= 2\cos\phi/2 \quad \text{where } \phi = \frac{2\pi d}{\lambda}\sin\Theta$$

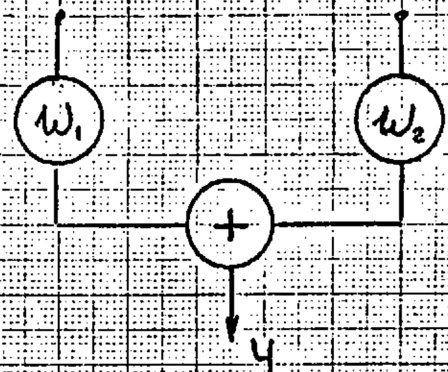
Example. If $d = \lambda/2$ $\phi = \pi \sin\theta$





Polar Coordinates

Figure 2. Two element array with null steering



$$y = w_1 e^{j\phi/2} + w_2 e^{-j\phi/2}$$

Constraint $y = 1$ when $\phi = 0$

Thus $w_1 + w_2 = 1$

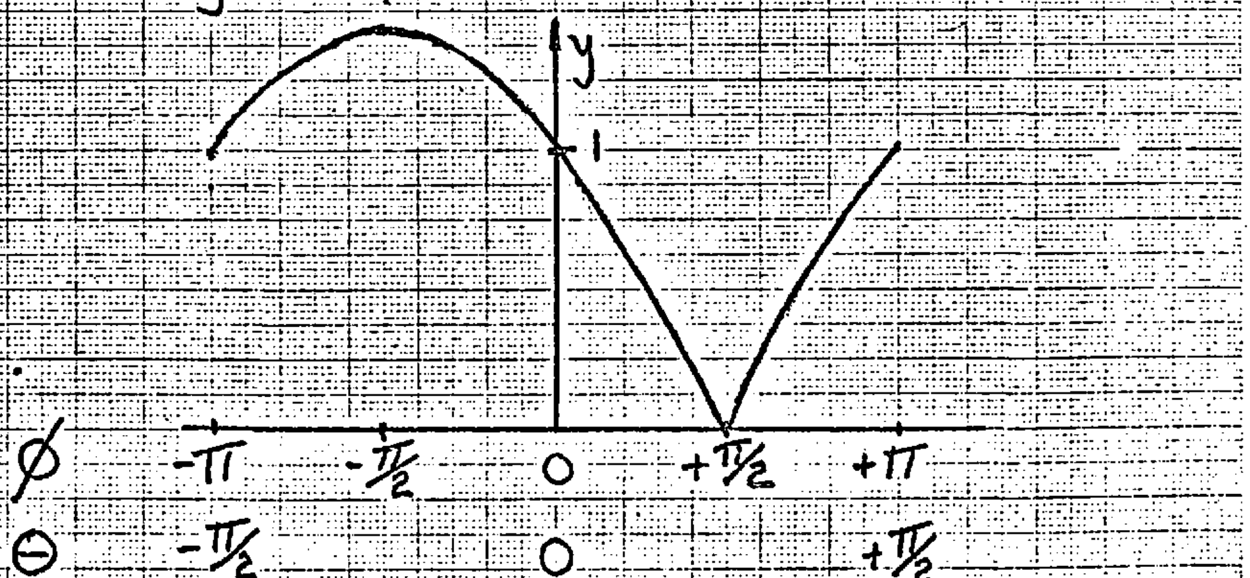
For null at say ϕ_1

$$w_1 e^{j\phi_1/2} + w_2 e^{-j\phi_1/2} = 0$$

Hence $w_1 = \dfrac{1}{1 - e^{j\phi_1}}$ + $w_2 = \dfrac{1}{1 - e^{-j\phi_1}}$

If $\phi_1 = \pi/2$ $\quad w_1 = \frac{1}{2}(1+j) \quad w_2 = \frac{1}{2}(1-j)$

$$y = \cos\phi/2 - \sin\phi/2 = \sqrt{2}\cos\left\{\phi/2 + \pi/4\right\}$$



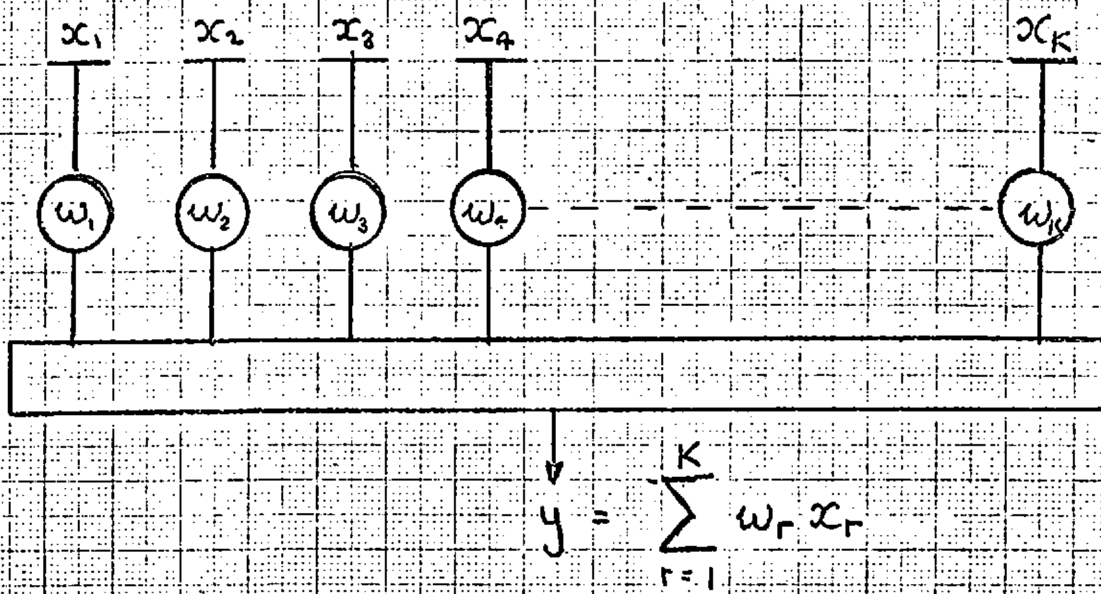


Figure 3 Multi-element Array

Figure 4 **Matrix formulation**

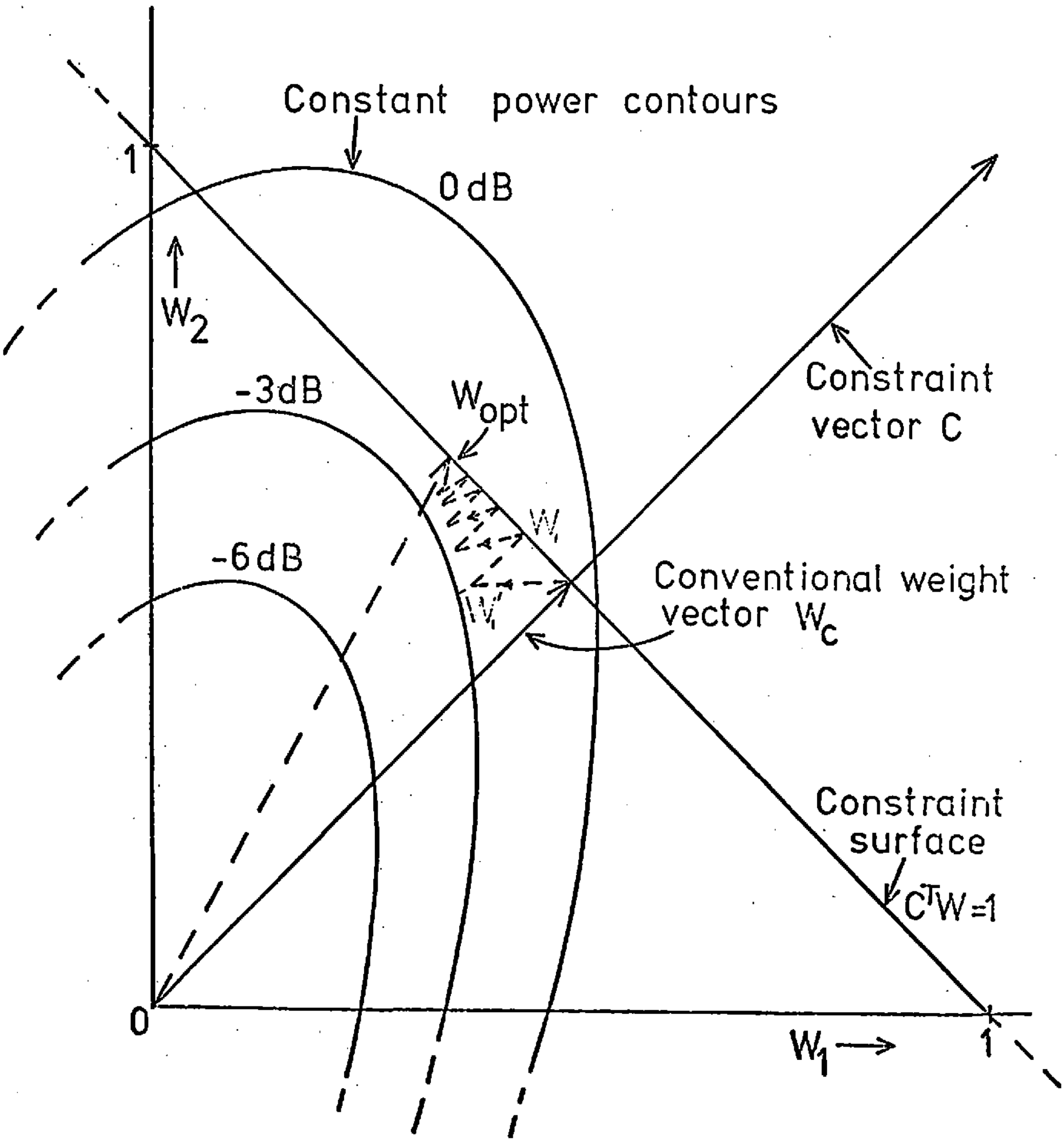
$$S = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_K \end{bmatrix} \quad N = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_K \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ \vdots \\ x_K \end{bmatrix} \quad W = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_K \end{bmatrix}$$

$$X = S + N \qquad y = W^T X$$

We wish to minimise $E(y^2)$ with the constraint

$$\sum_{i=1}^{K} w_i s_i = 1 \quad \text{i.e.,} \quad W^T S = 1$$

Figure 5. Simple case showing equal power contours



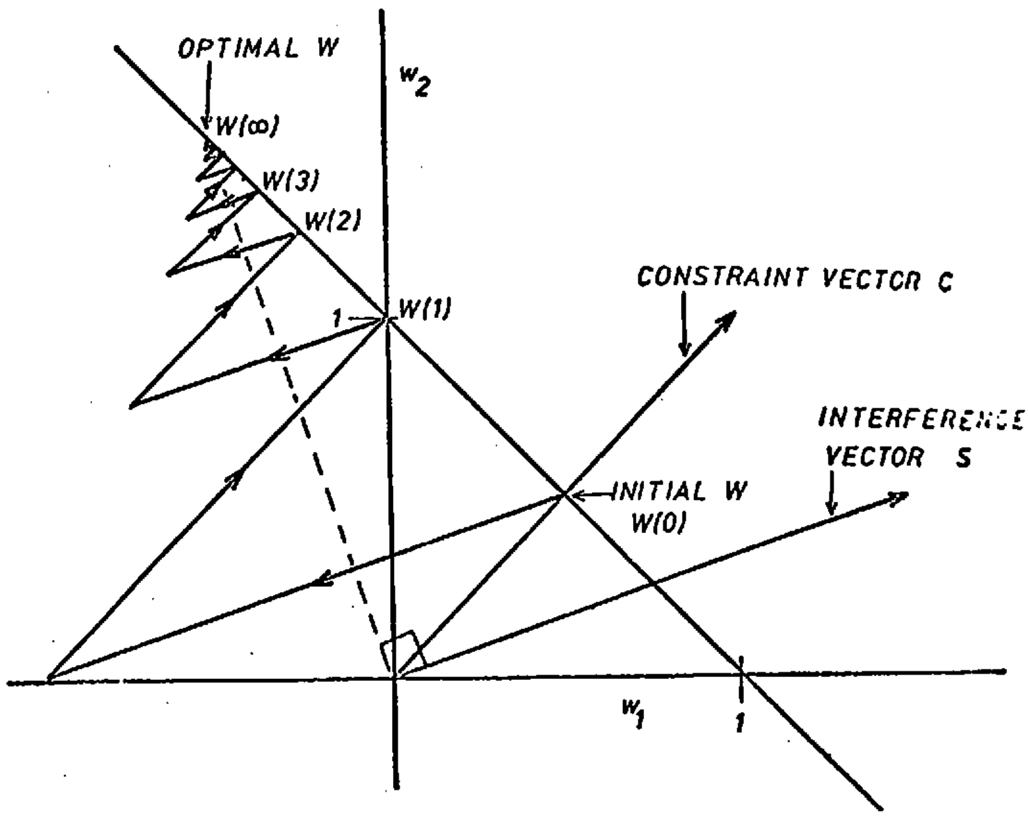


Figure 6. Simple case showing iterative solution

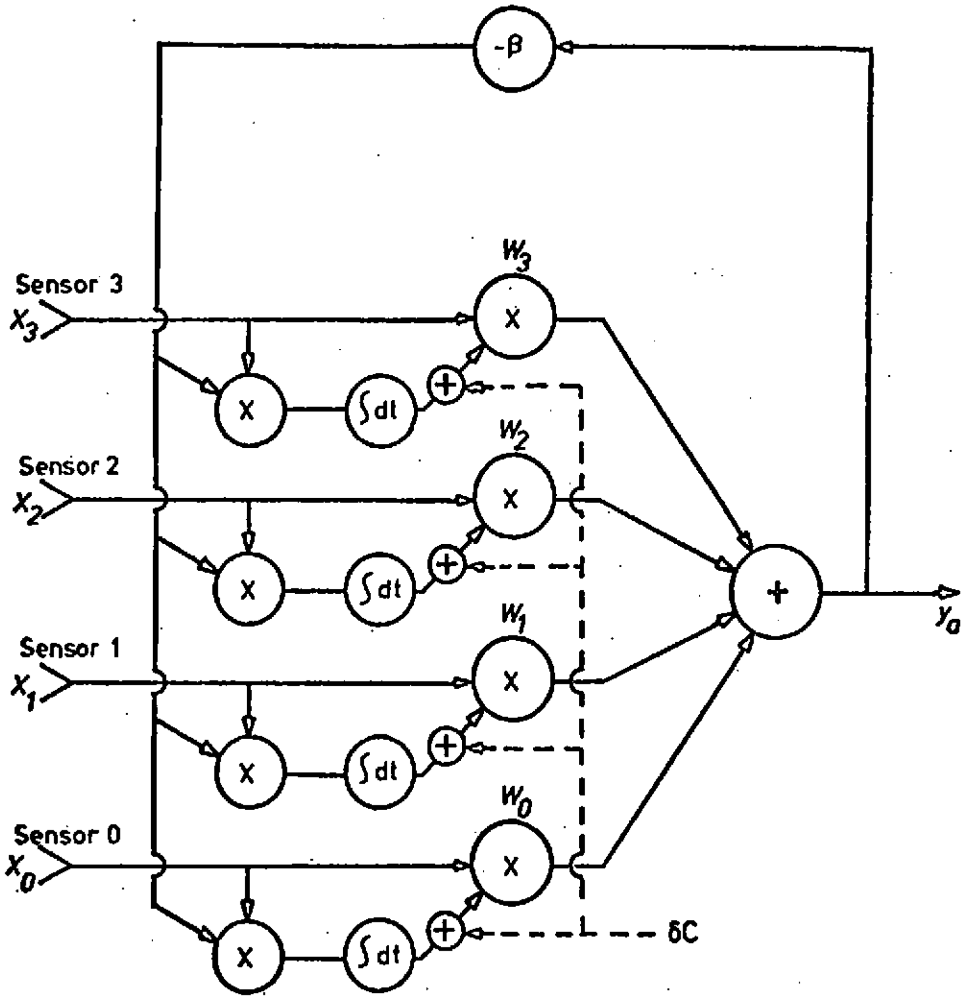


Fig. 7 A Narrow-Band Array-Space Maximum likelihood (Frost) circuit.