Proceedings of The Institute of Acoustics

THE TUNING OF KEYBOARD INSTRUMENTS

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Introduction

In view of the present interest in authentic performances of Renaissance and Baroque music, the author designed and had constructed an electronic organ (1) which could be instantly tuned to various obsolete systems of tuning as well as to the modern Equal Temperament (E.T.). A tape-recording illustrating the sounds of these systems has been made. The present paper is an adjunct to this demonstration, summarising the historical reasons for the changes through the centuries and presenting in diagrammatic form the principles behind the main systems of tuning.

The set of unequally-spaced notes sounded by the white keys of a piano (the diatonic series) has been evolved independently in many parts of the world. This is probably due to its being founded on two intervals (frequency-ratios), the octave (2:1) and the fifth (3:2) which are the first two intervals sounded by a primitive horn. Next to the unison they are the easiest intervals to tune accurately by the absence of beats between harmonics.

The Pythagorean scale

Pythagoras (c. 500 B.C.) standardized the method of tuning the diatonic series using perfect octaves and fifths only; the results are illustrated in Figure 1, the tuning sequence (using modern notation) being: C, G, D, A, E, B followed by C, C^1, F. The interval of a perfect fourth (4:3) can be used (e.g. G, D) because a downward fourth leads to the same note as an upward fifth followed by a downward octave. The sequence of tones and semitones is shown by the

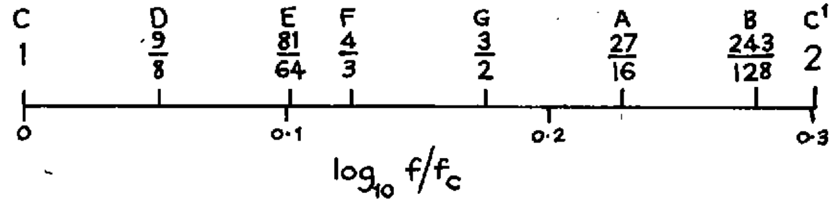

Figure 1. The Pythagorean scale

logarithmic scale in Figure 1. The Pythagorean scale was used in European music for nearly 2000 years, mainly for singing or playing in unison or at the octave or the fifth. However, its imperfections were realized from the fact that if the upper tonic (C^1) is tuned from the lower tonic (C) in a series of 12 fifths and fourths using the hypothetical "black notes" not then in general use, the upper tonic is found to be noticeably sharp when compared directly with the lower tonic. The discrepancy, known as the "comma of Pythagoras", originates in the fact that

$$(3/2)^{12} \neq 2^7 \quad \text{(i.e. } 129{\cdot}74 \neq 128\text{)}$$

The modern equal temperament system avoids this difficulty by defining the frequency-ratio of the semitone interval as the twelfth root of 2 (1·0594631..). Figure 4 shows the divergence in pitch of the Pythagorean notes from E.T., the slope of the line being characteristic of the perfect fifth.

Zarlino's system

During the fifteenth century harmonic music incorporated the major and minor thirds (5:4 and 6:5) and their "complements", the minor and major sixths (8:5 and 5:3). The Pythagorean intervals are only rough approximations to these, as can be shown from the relative frequencies given in Figure 1. For example, the major third C-E would have about 16 beats per second in the middle frequencies. Zarlino (c. 1550) evolved a system (later called "just intonation") based on three perfect major triads, i.e. 3-note chords with frequency-ratios 4:5:6. Forming a sequence as shown in Figure 2, these triads (with octave transpositions) define a scale superficially like the Pythagorean but having all major thirds and, with the exception of one of each, all fifths and minor thirds in perfect consonance. Harmonization based on the tonic (C), dominant (G) and subdominant (F) triads and their inversions can be played without any "beating" whatever. The minor triad on D, however, is far from perfect, the fifth and the minor third being both dissonant.

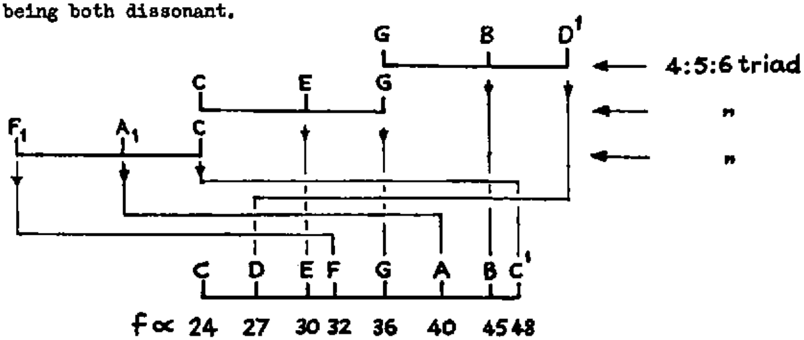

Figure 2. The Zarlino scale

One of the relatively new features of music of this period was modulation, the transition to a diatonic series starting on a note other than C. This required the provision of extra finger-keys, e.g. one between F and G which would give a diatonic series starting on G. We now (confusingly) call this scale a new "key". In Zarlino's system this simplest of all modulations promoted the bad fifth to a position in the important new dominant triad. Further difficulties arose with other modulations and it became clear about 1600 that a more flexible system of tuning was needed. Figure 5 illustrates the properties of the Zarlino scale in relation to E.T. The slopes of the ascending lines again characterize the perfect fifths in this scale.

The mean tone system

The addition of five black keys to the keyboard provided the means (at least in principle) of modulating freely amongst 12 "keys". Musicians in the 16th century, including the Englishman John Bull, had already composed pieces passing through all 12 "keys" and back to the starting "key", a procedure suggestive of a journey around the "keys" arranged like the hours of a clock-face as in Figure 3. Although equal temperament was known in principle at this time the harshness of its major and minor thirds (10 to 20 beats per second in mid-keyboard) were unacceptable to ears accustomed to the smooth intervals of Zarlino's system.

The "mean tone" tuning which emerged at this time offers perfect major thirds in a range of adjacent "keys" by the use of shortened fifths. Minor thirds were consequently also slightly short, though less distorted than the minor thirds in E.T. Organs were tuned on this system from the early 17th century to the mid-eighteenth. The sequence of tuning is illustrated in Figure 3 in which the 12 equally-spaced divisions around the outside of the circle represent E.T. tuning. Starting at C, the notes G, D, A and E are tuned in sequence by fifths appropriately shortened (and fourths lengthened) to bring the frequency of E to exactly 5/4 that of C. The shortening of the fifths is shown (exaggerated) inside the circle in Figure 3. Tuning proceeds similarly from E through B, F# and C# to G#, an exact major third above E. If this procedure were continued it would lead to an intolerably flat C^1, since three major thirds represent a frequency-ratio of

$$(5/4)^3 = 125/64$$

which is significantly less than 2. Instead, tuning proceeds in the opposite ("flatwards") direction starting at C^1 which has been tuned to an exact octave above C. Again, shortened fifths and lengthened fourths are used in the sequence C^1, F, Bb, Eb, progress to Ab being stopped by the fact that this finger-key has already been used for G#. Some organs in England had the finger-key divided so that both G# and Ab could be played, Ab being considerably higher in pitch than G#. The errors resulting from the use of shortened fifths accumulate in the interval G# to Eb^1, the "wolf fifth". In this system all major thirds are perfect and all minor thirds nearly so with key-signatures for Bb, F, C, G, D and A. All fifths are nearly perfect (slightly less so than in E.T., however) with these key-signatures. The remaining key-signatures are all subject to various degrees of discord for triadic harmony. Further restrictions are set by the accidentals in the modern minor modes, but in spite of this the mean-tone system allowed phenomenal development of organs and of organ music in the 17th century. The main features of the system are summarized in Figure 6. The falling slope characterizes perfect major thirds.

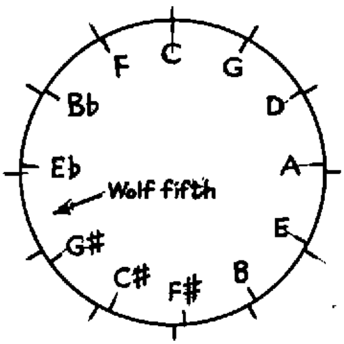

Figure 3. The mean tone system

The long reign of the mean-tone system is probably connected with the custom (especially in northern Europe) of providing organs with powerful ranks of pipes (tierce, sesquialtera etc.) sounding the true fifth harmonic of the note played. It is common for a bass note and another two octaves and a major third higher to be played together. If the latter is not tuned as an exact major third there are loud beats between it and the tierce pipe sounded by the bass note, a defect which is accepted now (even in "authentic" circles) but which was thought to be inadmissible in the 17th century.

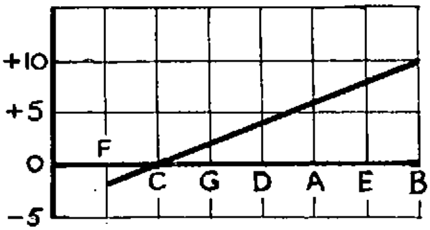

Figure 4. Deviation (in cents) of the Pythagorean scale from E.T.

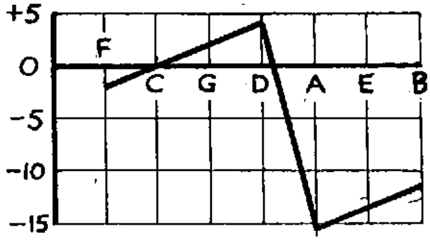

Figure 5. Deviation (in cents) of the Zarlino scale from E.T.

(100 cents = 1 semitone E.T.)

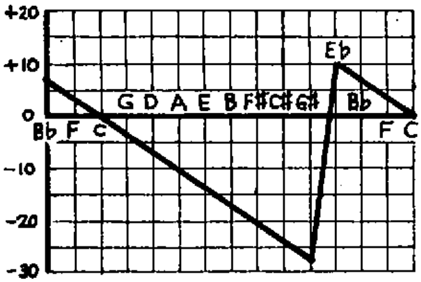

Figure 6. Deviation (in cents) of the mean tone scale from E.T.

Figure 7. Deviation (in cents) of the Werckmeister scale from E.T.

(100 cents = 1 semitone E.T.)

The Werckmeister system

Bach's advocacy of the "wohltemperierte Klavier" was not a plea for E.T., which did not become common until the late 19th century, but for some modification (tempering) of the mean tone system to allow freedom of modulation without suppressing all differences between "keys" as in the E.T. system. The Werckmeister system is one such. As can be seen from Figure 7, Werckmeister distributed the imperfections in a manner approaching E.T. but with the major thirds generally better in the sharp "keys" than in the flat ones, the converse being true for the fifths and minor thirds.

Reference: (1) K.A.MACFADYEN and DAVID GREER,1969 Musical Times 110, 623-4
A detunable organ.