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THE DEVELOPMENT OF A WAVE APPROACH TO STATISTICAL ENERGY ANALYSIS

Dr K. H. Heron

Royal Aerospace Establishment, Materials and Structures Department,
Farnborough, Hampshire, GU14 6TD, ENGLAND

1. INTRODUCTION

Statistical Energy Analysis has traditionally been developed using a modal summation and averaging approach, and this has led to the need for the many well known and restrictive SEA assumptions. The assumption of 'weak coupling' is particularly unacceptable when attempts are made to apply SEA to structural couplings. Many scientists believe that this major assumption is more a function of the modal approach than a necessary requirement of SEA itself.

This paper ignores this restriction, and describes a wave approach to the calculation of beam-beam coupling loss factors. It is based on a calculation of the transmission matrix at a junction of semi-infinite beams, with each beam having four wave-types associated with it (longitudinal, torsional and two bending). The method assumes a point connection between the various beams but takes full account of beam orientation and the detailed geometry of the junction. The assumptions involved in using these transmission coefficients to obtain coupling loss factors are discussed. Also described is a parallel and novel wave approach to the deterministic prediction of the response of a beam network in which the SEA result is simply the first term of an infinite series.

Finally by considering individual modes the spread of exact results about the SEA prediction is discussed and simple formulae derived for this spread when the modal overlap factor is small.

2. BASIC BEAM EQUATIONS

2.1 Equations of motion

Practically all textbooks on the subject of beams either deal only with the static equations or include the time variant terms but make simplifying assumptions about the beam. In particular most authors assume that the beam shear axis and the beam centroid axis are coincident, though this is only true for doubly symmetric beam sections. Taking full account of such offsets and including the usual 'thick' bending terms, the 12 cyclic equations of motion developed with respect to the centroid axes are:

$$\partial F_x/\partial t = -ES\partial V_x/\partial x\,, \tag{2.1}$$

$$-W_z + (\chi_z/GS)\partial F_y/\partial t = -\partial(V_y - \varepsilon_z W_x)/\partial x\,, \tag{2.2}$$

$$W_y + (\chi_y/GS)\partial F_z/\partial t = -\partial(V_z + \varepsilon_y W_x)/\partial x\,, \tag{2.3}$$

$$\partial(M_x + \varepsilon_z F_y - \varepsilon_y F_z)/\partial t = -GJ\partial W_x/\partial x\,, \tag{2.4}$$

$$\partial M_y/\partial t = -B_y\partial W_y/\partial x\,, \tag{2.5}$$

$$\partial M_z/\partial t = -B_z\partial W_z/\partial x\,, \tag{2.6}$$

$$m\partial V_x/\partial t = -\partial F_x/\partial x\,, \tag{2.7}$$

$$m\partial V_y/\partial t = -\partial F_y/\partial x\,, \tag{2.8}$$

$$m\partial V_z/\partial t = -\partial F_z/\partial x\,, \tag{2.9}$$

$$H_x\partial W_x/\partial t = -\partial M_x/\partial x\,, \tag{2.10}$$

$$-F_z + H_y \partial W_y/\partial t = -\partial M_y/\partial x, \qquad (2.11)$$
$$F_y + H_z \partial W_z/\partial t = -\partial M_z/\partial x, \qquad (2.12)$$

The beam is assumed to lie on the x-axis and most of the symbols are self explanatory, V represents linear velocity, W rotational velocity, F force, and M moment. ε_y and ε_z are the coordinates of the shear axis with respect to the centroid axis, and χ_y and χ_z are the thick beam bending shear coefficients.

2.2 General solution

Without loss of generality we may look for solutions where all 12 field variables have the factor $e^{-\lambda x}e^{i\omega t}$, where ω is the radian frequency of interest and λ is as yet unknown.

Define a field variable column vector $\mathbf{q}$ such that its transpose is given by

$$\mathbf{q}^t = (V_x, V_y, V_z, W_x, W_y, W_z, F_x, F_y, F_z, M_x, M_y, M_z), \qquad (2.13)$$

where we have dropped the factor $e^{-\lambda x}e^{i\omega t}$. Then equations (2.1) to (2.12) can be rewritten in matrix form as

$$\mathbf{Cq} = \lambda\mathbf{q}, \qquad (2.14)$$

where $\mathbf{C}$ is a straightforward algebraic 12×12 complex matrix. Equation (2.14) can be solved to give 12 eigenvalues, λ_j say, and the 12 associated eigenvectors, $\mathbf{q}_j$ say. The general solution for the beam can then be written as

$$\mathbf{q}(x) = \mathbf{UT}(x)\mathbf{a}, \qquad (2.15)$$

where $\mathbf{U}$ is a matrix formed from the eigenvectors such that column j of $\mathbf{U}$ is $\mathbf{q}_j$, $\mathbf{T}(x)$ is a diagonal matrix formed from the eigenvalues such that diagonal element j is $\exp(-\lambda_j x)$, and $\mathbf{a}$ is a column vector of arbitrary constants dependent upon the beam boundary conditions.

3. COUPLING LOSS FACTORS

3.1 Point junction transmission coefficients

Consider a point junction made up from a collection of semi-infinite beams rigidly connected together at their ends. Using equation (2.15) the 6×6 semi-infinite point impedance matrix for each beam can be computed, and by applying suitable coordinate transformations to take account of beam orientations and beam axis offsets with respect to the junction, the full junction impedance matrix can be calculated by summing over all the beams connected at the junction. By imposing an incident wave of a particular type (*eg* torsion) on one of the beams, and then calculating the force and velocity this causes at the junction, it is then possible to calculate the transmission coefficient between the given input wave and any output wave on any beam. By using this process for each wave-type the complete transmission matrix can be calculated for any junction geometry provided only that the beams are rigidly connected at the junction. For a junction with two beams, for example, this matrix will be an 8×8 matrix to take full account of all four wave-types on each of the two beams. It can also be shown that this transmission matrix is always symmetric.

3.2 Assumptions needed for the wave approach to SEA

A basic assumption in the wave approach to SEA is that the transmission matrix described above for a junction of semi-infinite beams can be used at an equivalent junction on a finite beam network. This important assumption may initially appear very restrictive, but not if viewed from a physical point of view. If SEA is to work at all, then a basic concept is that under the various averaging processes generic to SEA the dependence on the precise length of a beam cannot be critical. This implies that the boundary condition at the end of the beams, remote from the junction under investigation, cannot be relevant. Thus we can arbitrarily choose these boundary conditions for mathematical convenience, and therefore our use of semi-infinite beams is fully justified. It is well worth noting that this imposes no limit on the use of SEA, unlike the modal approach with its 'weak coupling' requirement, rather it states that we should view an SEA prediction as some kind of mean prediction.

A further assumption is needed before we can proceed to the calculation of coupling loss factors. For each particular wave-type in each beam of the network, there will be power flowing along the beam in the two directions, and some relationship has to be assumed between these two power flows. There have been few attempts, with the notable exception of DeJong [1], to improve on the basic assumption due to Lyon [2] that states that these two power flows should be assumed equal. This assumption is directly equivalent to the random incidence assumption commonly made when dealing with plates and acoustic volumes. Of course a strict application of this assumption implies zero net power flow throughout the network, however we are not assuming strict equality, but only requiring them to be nearly equal. This latter point is analogous to the case of a reverberant room excited by a central source; away from the source near field we would not expect to have to modify our assumption of random incidence, even though the net outward power strictly invalidates this assumption.

3.3 Coupling loss factor calculations

Consider the power flowing from sub-system 1 to sub-system 2, and let Π_{1+} and Π_{1-} be the powers flowing in sub-system 1 towards and away from the junction respectively. Then if E_1 is the total energy in sub-system 1 and L is the length of sub-system 1

$$E_1/L = \Pi_{1+}/c_g + \Pi_{1-}/c_g\,, \tag{3.1}$$

where c_g is the group velocity associated with sub-system 1. Now by definition

$$\omega E_1 \eta_{12} = \Pi_{1+} \tau_{12}\,, \tag{3.2}$$

where η_{12} is the required coupling loss factor, and τ_{12} is the transmission coefficient (a single element of the transmission matrix described above). Combining equations (3.1) and (3.2) and assuming that Π_{1+} and Π_{1-} are equal as discussed above, we deduce that

$$\eta_{12} = \tau_{12}/2(\pi\omega n_1)\,, \tag{3.3}$$

where n_1 is the sub-system modal density given by

$$n_1 = L/(\pi c_g)\,. \tag{3.4}$$

Furthermore, since τ_{12} is equal to τ_{21},

$$n_1\eta_{12} = n_2\eta_{21}\,, \tag{3.5}$$

as expected.

4. SEA AND EXACT ANALYSIS

For the particular case of a point excited beam network we may view any theory as a prediction of the transfer mobility between the exciting point force and some response point. SEA can then be considered to be the first approximation of an infinite series of approximations which ultimately converge on the deterministic transfer mobility. We may track waves as they travel around the network, as they transmit and reflect at the junctions and as they decay along the beams. Normally such a wave approach to the deterministic result is simply terminated after a certain number of iteration cycles, that is junction transmissions, and any remaining wave power ignored. Of course if this residual power is significant its neglect will invalidate the result, thus the result after few iteration cycles is generally of no intrinsic value.

An alternative approach is to feed this residual power into an SEA model and add the result to the deterministic wave part of the prediction. Let M_0 be the straightforward SEA result, M_1 the result after one iteration cycle, M_2 the result after two iteration cycles, etc. M_1 is then the result obtained by calculating the amplitude of the point force induced waves, tracking these waves to the ends of the forced beams, *ie* to the first junction, then presenting the subsequent transmitted and reflected wave powers as power inputs to the SEA model, and finally adding the result from this SEA calculation to that obtained from the deterministic wave tracking along the drive beam. By this means the results after each iterative cycle are meaningful predictions. M_0 is the standard SEA result and M_∞ is the exact result.

By studying this series of solutions a better understanding of the SEA result might be obtained. For example, Fig 1 shows the results for a single beam in 'thin' bending, with the force at one end, and because of this special force position the standard SEA result, M_0, under-predicts, whereas M_1 is much better. Of course, in this case the result is not surprising and would be accommodated into standard SEA theory by allowing for force inputs at junctions, the curve labelled M_{junc} of Fig 1. However more complicated beam network theory may benefit from this approach, and it is certainly interesting to observe how the modal response characteristics appear in Fig 1.

5. VARIANCE

The likely variance of a particular result from the SEA prediction is very difficult to calculate; we are not much further forward today than the results published by Lyon [2] in 1975. This is mainly caused by the non-Gaussian form of the distribution of a set of particular results. Here we attempt to calculate the spread of expected results rather than their full statistical distribution. Firstly, whatever the forcing function, we can consider the ultimate aim to be the prediction of the response at some point. At frequencies associated with high mode numbers we must search for average responses and some measure of the expected spread of results from this average, hence the birth of SEA. But it must be remembered that initially we measure single deterministic response functions, even though we may subsequently average across a frequency band and/or take spatial and ensemble averages. If for a given structure instead of taking these averages we overplot a series of particular response functions, and then study this picture, we quickly come to the conclusion that we do not necessarily need the full statistical distribution, simple measures based on the likely spread could suffice.

With this in mind, we must ask ourselves what 'the SEA result' implies, particularly in regions of low modal overlap. In such a region we have well separated modes, and we can consider the SEA result to be the energy sum across these well separated modes within a wide frequency bandwidth, divided by this bandwidth. Assuming that all the modes have equal response levels, it can easily be shown that the SEA result for the response, R_{sea}, is given by

$$|R_{sea}|^2 = C\pi n/(2\omega\eta) \qquad (5.1)$$

where C is some constant, n is the system modal density, and η is the typical mode energy damping factor. The peak level, R_{peak}, which occurs at a system resonance is given by

$$|R_{peak}|^2 = C/(\omega\eta)^2 . \qquad (5.2)$$

Thus

$$|R_{peak}|^2/|R_{sea}|^2 = 2/\pi\theta\,, \tag{5.3}$$

where θ is the modal overlap factor, given by

$$\theta = \omega n\eta\,. \tag{5.4}$$

In order to compute the expected minima we must consider two adjacent modes, and we can either sum their responses and obtain an anti-resonance, or subtract them to obtain the typical shallow minimum. Letting these responses be designated R_{min+} and R_{min-} respectively, it is easy to show that

$$|R_{min+}|^2/|R_{sea}|^2 = 8\theta^3/\pi\,, \tag{5.5}$$

and

$$|R_{min-}|^2/|R_{sea}|^2 = 8\theta/\pi\,. \tag{5.6}$$

It is satisfying that all these formulae are simple functions of θ the modal overlap factor. Investigators of SEA variance distributions should perhaps look for functions involving only this one parameter.

Fig 2 is a plot of the results for a single beam in 'thin' bending. The beam is 0.03 m × 0.05 m × 3 m and has a fundamental bending frequency of 48 Hz, and nine deterministic transfer mobilities have been plotted associated with three point forces at distances 0.53 m, 1.02 m and 1.14 m from one end, and three accelerometer response points at distances 2.14 m, 2.48 m and 2.59 m from the same end. An energy damping factor of 0.02 was used. The four smooth prediction lines have been calculated using formulae (5.3), (5.5) and (5.6) in conjunction with the standard SEA prediction. The results for this very special case are excellent.

6. CONCLUSIONS

Formulae have been presented that allow a full SEA prediction of a general beam network. A novel part deterministic and part statistical approach has been outlined. A simple approach to the spread of particular results about the SEA prediction has been outlined based on the modal overlap factor.

7. REFERENCES

[1] R.G. DeJong, SEA seminar, Cambridge, MA., 1986 (*private communication*)
[2] R.H. Lyon, 'Statistical Energy Analysis of dynamical systems: Theory and application', Cambridge, MA, MIT Press (1984)

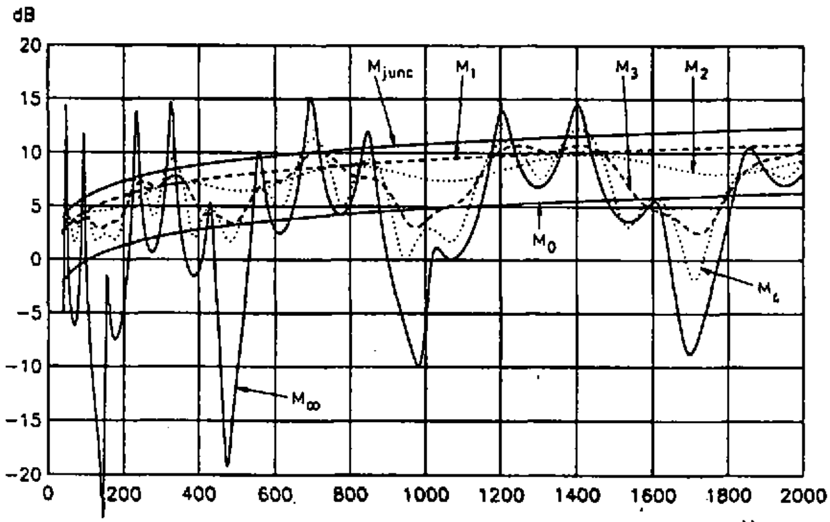

Fig 1 Single beam SEA and exact

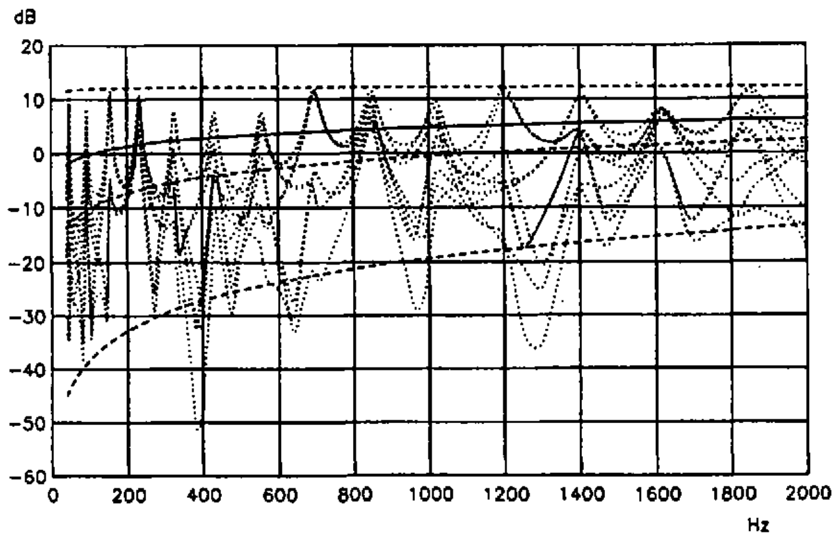

Fig 2 Single beam SEA variance