ON THE THEORY OF SOUND PROPAGATION IN A VELOCITY GRADIENT NEAR AN IMPEDANCE BOUNDARY

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Introduction

Although several numerical methods including Fast Field, Parabolic Equation and Normal Mode Solutions are available for predicting outdoor sound propagation over a flat finite impedance ground, ray trace methods offer greatest convenience both in terms of computation and physical insight. Traditional ray trace methods cannot allow properly for interaction with the ground. On the other hand a heuristic modification of the classical Weyl-Van der Pol formulation of the total field in the presence of the ground has been advocated for incorporation in ray tracing codes. By using a WKB method of approximation a closed form solution to propagation in an <u>arbitrary</u> sound velocity gradient has been obtained. In the limit of zero gradient the solution reduces to the standard form for a homogeneous atmosphere. If the velocity gradient is linear then it is shown that the solution may be reconciled with the Airy function form for propagation with upward refraction and the normal mode solution for downward refraction.

Theory

<u>(a) First order approximation</u>

Consider a point monopole source of angular frequency ω placed at $(0,0,z_s)$ in a stratified medium near an impedance boundary at $z = 0$. The speed of sound, c is no longer constant but varies with the vertical height z above the plane boundary. The wave number ω/c is denoted by k which is a function z only as a result of the variation of c in the stratified medium. Additionally, the boundary surface is assumed to be locally reacting and its specific normal admittance is given by β. See figure 1 for the source and receiver geometry.

The derivation of an approximate solution, for a field point situated at z_r above the impedance plane and a horizontal separation of r was given in ref. [1]. The result is essentially a high frequency approximation that is based on the ray theory analysis. The result may be summarised in the Weyl-Van der Pol form as follows.

$$p = S(\phi)\, e^{ik_0 R_1} / 4\pi\bar{R}_1 + S(\theta)\, Q\, e^{ik_0 R_2} / 4\pi\bar{R}_2 \qquad (1)$$

where $$R_1 = \int_{z_r}^{z_s} (n/\cos\phi)\, dz' \ ,$$

$$R_2 = \int_{0}^{z_r} (n/\cos\phi)\, dz' + \int_{0}^{z_s} (n/\cos\phi)\, dz' \ ,$$

$$\bar{R}_1 = \sqrt{r\, R_1/\sin\phi_o}$$

$$\bar{R}_2 = \sqrt{r\, R_2/\sin\theta_o}$$

$$Q = R_p + S_\beta\,(1 - R_p)F(w)\,/\,S(\theta)$$

$$R_p = \frac{\cos\theta_o - \beta}{\cos\theta_o + \beta} \ ;$$

$$F(w) = 1 + i\sqrt{\pi}\, w\, e^{w^2} \operatorname{erfc}(-iw) \ ;$$

$$w^2 = \frac{1}{2} i\, k_o R_2 \,(\cos\theta_o + \beta)^2$$

$$S(\mu) = \sqrt{\frac{\cos\mu_o}{n_r \cos\mu_r}}\sqrt{\frac{\cos\mu_o}{n_s \cos\mu_s}}$$

$$S_\beta = \beta\,/\,(1 - n_r^2 - \beta^2)^{\frac{1}{4}}\,(1 - n_s^2 - \beta^2)^{\frac{1}{4}} \ ,$$

where the subscripts r and s denote the properties at the source plane and receiver plane respectively.

We note that ϕ and θ are the elevation angles (measured from the vertical z-axis) for the direct and reflected waves respectively. The sign convention of $e^{-i\omega t}$ is understood , n is the index of refraction of the medium that is a arbitrary function of z only and the subscripts o and s denote the conditions at the ground level and the source plane respectively. Further the source height, z_s is assumed to be greater than the receiver height, z_r . The reciprocity theorem ensures that one can exchange the position of the source and receiver if $z_r > z_s$. Snell's Law is used to relate the elevation angle with the index of refraction by,

$$n \sin \mu = \sin \mu_o \quad ,$$

where μ is the elevation angle of a ray launched from the source.

It should be emphasised that equation (1), although an approximate solution for an arbitrary gradient, is only valid for the condition of a single reflection. This restriction means that our analysis is restricted to the sound field at relatively short ranges. The sound velocity gradient is assumed to be small such that there will be no shadow zones, and, more importantly, that there are no multiple ray paths for the direct and reflected waves in the area of interest. Consequently the analysis in ref. [1] may be applied in our present situation.

<u>(b) Higher order approximation for the reflection coefficient</u>

The expression for the total sound field given in equation (1) is a first approximation in which terms of the order of $1/k_o$ and above are ignored. The approximation for the reflection coefficient R_p becomes increasingly inadequate when the source and receiver are close to ground and the impedance is sufficiently high. This can be traced back to the approximation used in the solution of the transformed pressure wave equation,

$$\frac{d^2\hat{p}}{dz^2} + k_o^2 N^2 \hat{p} = -\delta(z-z_s) \qquad (2)$$

where

$$N = \sqrt{n^2 - \cos \mu_o}$$

$$= n \cos \mu \quad .$$

The transformed pressure is related to the solution for the sound pressure by

$$p = \frac{1}{2\pi}\int_0^{\infty} k\, J_o(k\, r)\, \hat{p}\, dk \quad .$$

The approximate solution used in Eq. (2) included terms in $\exp\{\pm ik_oL\}/\sqrt{N}$ which represent sets of two non-interacting waves, namely the outgoing wave and the incoming wave, with higher order terms being ignored. These terms can be expressed more accurately[2] by a power series of k_o in the form of $E\, e^{-ik_oL}/\sqrt{N}$ and $F\, e^{ik_oL}/\sqrt{N}$ respectively, with

$$E(k_o) = \sum_{q=0}^{\infty} (-1)^q \frac{A_q(z)}{(i\, k_o)^q} , \tag{3}$$

$$= 1 - \frac{A_1(z)}{ik_o} + \frac{A_2(z)}{(ik_o)^2} - \frac{A_3(z)}{(ik_o)^3} + \cdots\cdots$$

$$F(k_o) = \sum_{q=0}^{\infty} \frac{A_q(z)}{(i\, k_o)^q} , \tag{4}$$

$$= 1 + \frac{A_1(z)}{ik_o} + \frac{A_2(z)}{(ik_o)^2} + \frac{A_3(z)}{(ik_o)^3} + \cdots\cdots$$

We note that A_o is equal to 1.

$$L = \int_0^z N(\varsigma)\, d\varsigma , \tag{5}$$

$$A_{q+1}(z) = -\frac{1}{2}\frac{1}{N}\frac{dA_q}{dz} + \int \Lambda(z)\, A_q(z) dz \tag{6}$$

and
$$\Lambda(z) = -\frac{1}{2}\frac{1}{N^{3/2}}\frac{d^2}{dz^2}\left(\frac{1}{\sqrt{N}}\right) . \tag{7}$$

If the term of $(1/k_o)$ is included, the reflection coefficient can be recast as,

$$R = \frac{\cos\mu_o - \beta_a}{\cos\mu_o + \beta_a} \tag{8}$$

where β_a is the apparent admittance given by

$$\beta_a = \beta + i\left[\frac{(N')_o}{2N_o} - N_o A_1\right]/k_o \quad , \tag{9}$$

and A_1 represents $A_1(z)|_{z=0}$ and β_a tends to β at high frequencies. The second term of Eq. (9) may be regarded as a correction factor due to the presence of a sound velocity gradient.

(c) Application tool : A linear sound velocity profile

The foregoing development of ray theory in outdoor sound propagation was based on an arbitrary sound speed profile. It is apparent that the sound field depends critically on this profile in a vertically stratified medium, through Eq. (1). We can evaluate the path lengths and the angle of incidence provided that the sound speed profile is specified as a function of vertical height. It is convenient to consider a simple idealised situation where the speed of sound varies linearly with the vertical height.

There are two advantages of assuming a linear sound velocity profile. Firstly, the use of the linear profile leads to circular ray paths[3] and it is relatively easy to 'trace' the direct and reflected waves. Secondly, there is an exact theory of propagation in a linear gradient where the solution can be expressed in terms of Airy functions and their derivatives[4]. This solution can be used to compare with that result from the ray theory approximation presented here.

In a stratified medium with a linear sound velocity profile, the speed of sound c and the index of refraction n are simply given by

$$c = c_o(1 + a z)$$

and

$$n = \frac{1}{1 + az} \quad .$$

where a is the normalised sound velocity gradient given by

$$a = \frac{1}{c_o}\left(\frac{dc}{dz}\right) \quad .$$

In general, the normalised sound velocity gradient is small such that N may be approximated by

$$N \approx \sqrt{(1 - 2az) - \sin^2\mu_o} \quad . \tag{10}$$

We note that a negative sound velocity gradient corresponds to an upward refracting medium, whilst a positive value corresponds to a downward refracting medium. The solution of Eq. (2) can be expressed as

$$\hat{p} = \frac{i}{2k_0\sqrt{N_s}}\left\{\frac{\exp[ik_0(L_s - L_r)]}{\sqrt{N_r}} + R\,\frac{\exp[ik_0(L_s + L_r)]}{\sqrt{N_r}}\right\} \quad . \tag{11}$$

It is obvious from Eq. (10) that

$$N_0 = \cos\mu_0 \qquad \text{and} \qquad N_0' = 1/\cos\mu_0 \quad . \tag{12}$$

Using Eqs. (5) and (10), L_r and L_s can be easily calculated. Integration shows that

$$L_r = \frac{1}{3a}\{\cos^3\mu_0 - [\cos^2\mu_0 - 2az_r]^{3/2}\} \tag{13}$$

and

$$L_s = \frac{1}{3a}\{\cos^3\mu_0 - [\cos^2\mu_0 - 2az_s]^{3/2}\} \quad . \tag{14}$$

Further A_1 can be evaluated, by using Eq. (6), to give

$$A_1 = -5a/(24\cos^3\mu_0) \quad . \tag{15}$$

Substitution of Eqs. (12) and (15) into (9) leads to

$$\beta_a = \beta - 7ia/(24k_0\cos^2\mu_0) \quad . \tag{16}$$

The transformed pressure given in Eq. (11) can be recast as

$$\hat{p} = \frac{i\exp\{ik_0[\cos^3\mu_0 - (\cos^2\mu_0 - 2az_s)^{3/2}]/3a\}}{2k_0[\cos^2\mu_0 - 2az_s]^{1/4}} \times$$

$$\left\{\frac{\exp\{-ik_0[\cos^3\mu_0 - (\cos^2\mu_0 - 2az)^{3/2}]/3a\}}{[\cos^2\mu_0 - 2az]^{1/4}} + \right.$$

$$\left.\frac{\cos\mu_0 - \beta_a}{\cos\mu_0 + \beta_a} \times \frac{\exp\{ik_0[\cos^3\mu_0 - (\cos^2\mu_0 - 2az)^{3/2}]/3a\}}{[\cos^2\mu_0 - 2az]^{1/4}}\right\} , \tag{17}$$

where β_a, the apparent admittance, is given by Eq. (16). Equation (17) represents an approximate solution for the transformed pressure in a linear sound velocity gradient above an impedance plane. We now outline an exact solution for this quantity.

The reduced wave equation, where N is approximated by Eq. (10), has a solution in terms of the Airy functions and their derivatives.[4] By imposing the boundary condition at $z = 0$, the particle velocity discontinuity and pressure continuity condition at the source plane, and the Sommerfeld radiation condition at infinity, the solution of Eq. (2) can be written as,

$$\hat{p} = -2\pi\,(2a\,k_0^2)\,e^{i\pi/6}\,\mathrm{Ai}[\,-(k_0/2a)^{2/3}(\cos^2\mu_0 - 2az_s)\,] \times$$

$$\{\mathrm{Ai}[\,-(-k_0/2a)^{2/3}\,(\cos^2\mu_0 - 2az)\,] - \Gamma(\mu_0)\,\mathrm{Ai}[\,-(k_0/2a)^{2/3}\,(\cos^2\mu_0 - 2az)\,]\,\}. \qquad (18)$$

The first and second terms of the above equation may be identified as the direct and reflected waves respectively and $\Gamma(\mu_0)$ is defined as the reflection factor[5] as follows,

$$\Gamma(\mu_0) = \frac{(-k_0/2a)^{2/3}\,\mathrm{Ai}'[-(-k_0/2a)^{2/3}\cos^2\mu_0] + (ik_0\beta/2a)\,\mathrm{Ai}[-(-k_0/2a)^{2/3}\cos^2\mu_0]}{(k_0/2a)^{2/3}\,\mathrm{Ai}'[-(k_0/2a)^{2/3}\cos^2\mu_0] + (ik_0\beta/2a)\,\mathrm{Ai}[-(k_0/2a)^{2/3}\cos^2\mu_0]}. \qquad (19)$$

The primes in Eq. (19) are their derivatives with respect to their arguments.

Principal values should be chosen for the complex roots in Eqs. (18) and (19). It is interesting to note that separate expressions, as suggested by Rasmussen[6], are not required for a different sign of the sound velocity gradient. Equation (19) is an unified expression that can be reduced to the form used by ,for example, Raspet *et. al.*[5] for a positive sound velocity gradient and to that used by, for example, Daigle and Berry[7] and Raspet *et. al.*[8] for a negative gradient.

The transformed pressure can be expanded in its asymptotic form by using the following asymptotic expansions for the Airy function and its derivative[9],

$$\mathrm{Ai}(z) \approx \frac{1}{2}\pi^{-1/2} z^{-1/4} e^{-\zeta} \sum_{0}^{\infty} (-1)^k c_k \zeta^{-k}$$

and

$$\mathrm{Ai}'(z) \approx \frac{-1}{2}\pi^{-1/2} z^{1/4} e^{-\zeta} \sum_{0}^{\infty} (-1)^k d_k \zeta^{-k}$$

where $\zeta = \frac{2}{3} z^{3/2}$,

$c_0 = 1 \; ; \; d_0 = 1$,

$c_k = (2k+1)(2k+3)\ldots(6k-1)/216^k \, k!$,

$d_k = -c_k \, (6k+1)/(6k-1)$,

$k = 1, 2, 3, \ldots$

and $|\arg z| < \pi$.

After some algebraic manipulations, it is not difficult to identify the reflection coefficient from the reflection factor $\Gamma(\mu_0)$,

$$R = \frac{(1 - d_1 \zeta_1^{-1}) \cos \mu_0 - \beta}{(1 - d_1 \zeta_2^{-1}) \cos \mu_0 + \beta} + 0\,(1/k_o^2) \; , \tag{20}$$

where $\zeta_1 = (-i \, k_0 \cos^3 \mu_0 / 3a)$,

$\zeta_2 = (i \, k_0 \cos^3 \mu_0 / 3a)$.

By making use the definition of c_k and d_k, we can show that Eqs. (8) and (19) are identical expressions for the reflection coefficient. In addition, it can be checked that the transformed pressure $\hat{p}$ given in Eqs. (17) and (18) are identical. It is reassuring to start with an exact analysis and to end with the same expression as the ray theory approximation.

Concluding Remarks

A WKB method of approximation has been used to derive a computationally straightforward solution for propagation in an arbitrary sound velocity profile above an impedance plane. To improve this solution for high impedance and grazing incidents a second order approximation for the reflection coefficient has been obtained. This introduces an effective admittance for the boundary which depends on the index of refraction at the ground.

For the case of a linear sound velocity profile the ray based approximation has been shown to be identical to the asymptotic form of the exact solution in terms of Airy functions. The latter has been presented in a form which is valid for either upwards or downwards refraction.

Current work is concerned with establishing the accuracy of the ray based approximations by comparing them with full wave numerical solutions and with experimental results.

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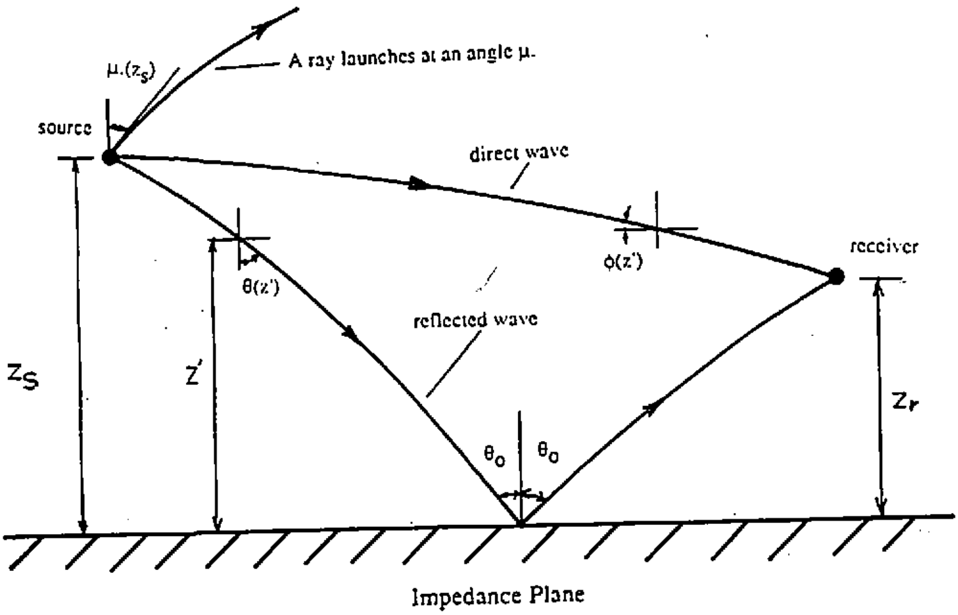

Figure 1 : The source/receiver geometry in a stratified medium.