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THE EXACT CALCULATION OF QUADRUPOLE SOURCES FOR SOME INCOMPRESSIBLE FLOWS

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INTRODUCTION

The acoustic analogy of Lighthill *[1]* and in particular its application to sound generated by surfaces in arbitrary motion by Ffowcs Williams and Hawkings *[2]* has been an extremely useful tool in estimating the aerodynamic sound generated by propellors and rotors. Although the Ffowcs Williams–Hawkings (FW-H) equation is intended for the prediction of the acoustic field given the aerodynamic field around the body, i.e. the *acoustic analogy*, the equation is an exact rearrangement of the mass and momentum conservation equations and can be used to recover the *aerodynamic* field near the body which is generating the sound as well. This relatively new idea has been attempted by Farassat and Myers *[3]*, Long *[4]*, and others.

In general practice, only some abbreviated form of the FW-H equation is used. One approximation to the FW-H equation which has often been applied for both acoustic and aerodynamic work is one in which the quadrupole source term has been ignored. It has been argued that the quadrupole term may be neglected for certian conditions for which the turbulent flow region is small *[5]*, however, probably the most fundamental reason it is left out is because it requires a detailed knowledge of the flow field around the body in advance. Without determining the entire flow field, and quite possibly the desired acoustic quantity, the volume necessary to adequately describe the quadrupole source is unknown, although reasonable guesses can be made. None-the-less, difficulty in obtaining a source term is little justification for the neglecting that term. Indeed Hanson and Fink *[6]* as well as Schimtz and Yu *[7]* have shown for high speed rotating blades that the quadrupole source is very important even though good results can be achieved in other operating ranges without the quadrupole.

In an effort to gain a new understanding about the quadrupole in both acoustic and aerodynamic applications, some sample problems have been chosen for which the flow field can be determined analytically using the two dimensional velocity potential. In the case of the circular cylinder, each of the source terms are calculated separately and compared with the exact potential solution. The forces on the cylinder due to pressure are compared as well. This problem helps to explain the results and difficulties of Brandão*[8,9]*.

The circular cylinder solution suggests a new description of the quadrupole term which is useful in identifying the volume and surface terms immediately from the exact solution. This is applied directly to find the relative source contributions for a Joukowski airfoil. Following this, the problem of a circular cylinder moving near a vortex filament is examined as well. Each of these cases illustrate the various roles of the volume source terms for incompressible flows. Another consideration of the role of quadrupole sources for exact *compressible* flow problems is given by Ffowcs Williams *[10]*.

PROBLEMS WITH EXACT SOLUTIONS

<u>The Circular Cylinder</u>

One of the most well know exact potential flow solutions is that for a circular cylinder in an inviscid, incompressible flow. This is such an important flow because the solution can be extended to a variety of other problems using conformal mapping of the complex velocity potential. Similarly, if one can understand the components of the flow as given by the FW-H equation, there is hope that these results can be transformed to give some idea to the behavior of each source term for a Joukowski

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airfoil. Indeed this has essentially been done in this paper. Brandão [8,9] has also used the circular cylinder problem in his development of an aerodynamic theory based on the FW-H equation, so comparisions can be made with his results.

Velocity Potential Solution– The velocity potential for a circular cylinder of radius a, in a frame of reference in which the cylinder is moving, is unsteady and known to be

$$\phi(\mathbf{x},t) = -\frac{a^2}{r}\mathbf{v}(t)\cdot\hat{\mathbf{r}} - \frac{K\theta}{2\pi} \tag{1}$$

where r,θ are the polar coordinates of $\mathbf{x}$, $\mathbf{v}(t)$ is the velocity of the cylinder center, K is the bound circulation on the cylinder and $\hat{\mathbf{r}}$ is a unit vector in the $\mathbf{x}$ direction. The perturbation pressure, given by the Bernoulli equation, is then written

$$p' = p - p_\infty = -\frac{1}{2}\rho u^2 - \rho\frac{d\phi}{dt} \tag{2}$$

where

$$u^2 = |\nabla\phi|^2 = \frac{a^4}{r^4}v^2 + \frac{Ka^2}{\pi r^3}v_t + \frac{K^2}{4\pi^2 r^2} \tag{3}$$

$$\frac{d\phi}{dt} = -\frac{a^2}{r^2}\{v_n^2 - v_t^2\} + \frac{Kv_t}{2\pi r} - \frac{a^2}{r}\frac{d\mathbf{v}}{dt}\cdot\hat{\mathbf{r}} \tag{4}$$

Here $v_n = \mathbf{v}\cdot\hat{\mathbf{r}}$ and $v_t = |\mathbf{v}\times\hat{\mathbf{r}}|$ since $\hat{\mathbf{r}}$ is also an outward unit normal vector to the surface. The terms are written out so that they may be compared with the solution gained from the FW-H equation.

Acoustic Solution– The Ffowcs Williams-Hawkings equation may be written

$$\nabla^2\{p'H(f)\} = -\frac{\partial^2}{\partial x_i\partial x_j}(\rho u_i u_j H(f)) + \frac{\partial}{\partial x_i}\{p' n_j\delta(f)\} - \frac{\partial}{\partial t}\{\rho_o v_n\delta(f)\} \tag{5}$$

for an inviscid, incompressible flow and where the derivatives are assumed to be generalized, $H(f)$ and $\delta(f)$ are the the Heaviside and Dirac delta functions respectively, and the three source terms are known as quadrupole source, loading source, and thickness source terms respectively. The function $f = 0$ is an equation which describes the body surface and shall be defined such that $\nabla f = \hat{\mathbf{n}}$, which is the outward unit normal vector.

The solution can be obtained using the Green's function for the Laplace equation and since the exact solution for pressure and velocity are known and the geometry is simple, each Green's function integral can be calculated analytically. When this is done, the pressures obtained are written

$$p'_t = \frac{\rho}{4}\left\{\frac{a^2}{r^2}(v_n^2 - v_t^2) + \frac{a^2}{r}\frac{d\mathbf{v}}{dt}\cdot\hat{\mathbf{r}}\right\} \tag{6}$$

$$p'_l = \frac{\rho}{4}\left\{\frac{a^2}{r^2}(v_n^2 - v_t^2) + \frac{a^2}{r}\frac{d\mathbf{v}}{dt}\cdot\hat{\mathbf{r}} - \frac{2Kv_t}{\pi r}\right\} \tag{7}$$

and

$$p'_q = -\frac{\rho}{2}\left\{\frac{a^4}{r^4}v^2 + \frac{Ka^2}{\pi r^3}v_t + \frac{K^2}{4\pi^2 r^2}\right\} \tag{8}$$

Here the subscripts t, l, and q refer to the thickness, loading and quadrupole contributions, respectively. It is immediately clear when comparing equations (6-8) with the potential solution, equations (3,4), that the thickness and loading sources correspond exactly to $-\rho d\phi/dt$ and the quadrupole contribution corresponds to $-\frac{1}{2}\rho u^2$. This is an interesting finding and warrants further exploration to determine if this correspondence can be generalised.

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Forces on the Cylinder– Notice in equations (6-8) that the total far-field solution is given by the thickness and loading, however in the case with circulation, $K \neq 0$, the quadrupole contribution can be as important as the thickness term. The quadrupole serves to provide a near-field pressure correction to the thickness and loading pressures. Figure 1 shows the relative contributions of each of the source terms for a cylider with circulation.

The force on the cylinder can now be easily calculated by integrating the pressure over the cylinder surface. The force per unit length is the found to be

$$\mathbf{F} = \mathbf{F}_t + \mathbf{F}_l + \mathbf{F}_q \tag{9}$$

where

$$\mathbf{F}_t = -\frac{1}{2}\rho_c\frac{d\mathbf{v}}{dt} \tag{10}$$

$$\mathbf{F}_l = -\frac{1}{2}\rho_c\frac{d\mathbf{v}}{dt} + \frac{1}{2}\rho K(\mathbf{v}\times\hat{\mathbf{k}}) \tag{11}$$

$$\mathbf{F}_q = \frac{1}{2}\rho K(\mathbf{v}\times\hat{\mathbf{k}}) \tag{12}$$

Here $\rho_c = \rho\pi^2 a$ which is the virtual mass of the cylinder and $\hat{\mathbf{k}} = \hat{\mathbf{n}}\times\hat{\mathbf{t}}$. The force composed of $\mathbf{F}_t$ and the first term of $\mathbf{F}_l$ is due to and opposes the acceleration of the cylinder while the force composed of $\mathbf{F}_q$ and the second part of $\mathbf{F}_l$ is due to circulation. It is apparent that the force generated by acceleration of the cylinder is independent of the quadrupole, but one half of the force due to circulation is given by the quadrupole term. This implies that if the FW-H equation is to be used for aerodynamic calculations, the quadrupole may be important for steady lifting problems.

A New Quadrupole Description

Before a more definitive statement is made, let us first return to examine the way in which the quadrupole term was simply related to $\frac{1}{2}\rho u^2$. With no loss of generality the volume term in equation (5) can be rewritten

$$\frac{\partial^2}{\partial x_i\partial x_j}\{\rho u_i u_j H(f)\} = \nabla^2\{\frac{1}{2}\rho u^2 H(f)\} + \rho\nabla\cdot\{((\boldsymbol{\zeta}\times\mathbf{u} + \mathbf{u}\nabla\cdot\mathbf{u})H(f)\} + \rho\nabla\cdot\{(u_n\mathbf{u} - \frac{1}{2}u^2\hat{\mathbf{n}})\delta(f)\} \tag{13}$$

where $\boldsymbol{\zeta} = \nabla\times\mathbf{u}$, is the local vorticity of the fluid. The surface term arises from the generalised gradient of $H(f)$, $\hat{\mathbf{n}}\delta(f)$. The second term on the right hand side (RHS) is zero for an irrotational $(\nabla\times\mathbf{u} = 0)$, incompressible $(\nabla\cdot\mathbf{u} = 0)$ flow. This quadrupole expression explicitly separates the $\frac{1}{2}\rho u^2$ part from the other parts. It is also useful to rewrite the thickness term

$$\frac{\partial}{\partial t}\{\rho v_n\delta(f)\} = -\nabla\cdot\{\rho v_n\mathbf{v}\delta(f)\} + \frac{d\mathbf{v}}{dt}\cdot\hat{\mathbf{n}}\delta(f) \tag{14}$$

which puts the steady part of the thickness term in a form similar to part of the surface term in equation (13). The FW-H equation may now be written

$$\nabla^2\{p' H(f)\} = -\nabla^2\{\frac{1}{2}\rho u^2 H(f)\} - \nabla\cdot\{((\boldsymbol{\zeta}\times\mathbf{u})H(f)\} - \nabla\cdot\{\rho v_n(\mathbf{u}-\mathbf{v})\delta(f)\} + \nabla\cdot\{(p' + \frac{1}{2}\rho u^2)\hat{\mathbf{n}}\delta(f)\} - \frac{d\mathbf{v}}{dt}\cdot\hat{\mathbf{n}}\delta(f) \tag{15}$$

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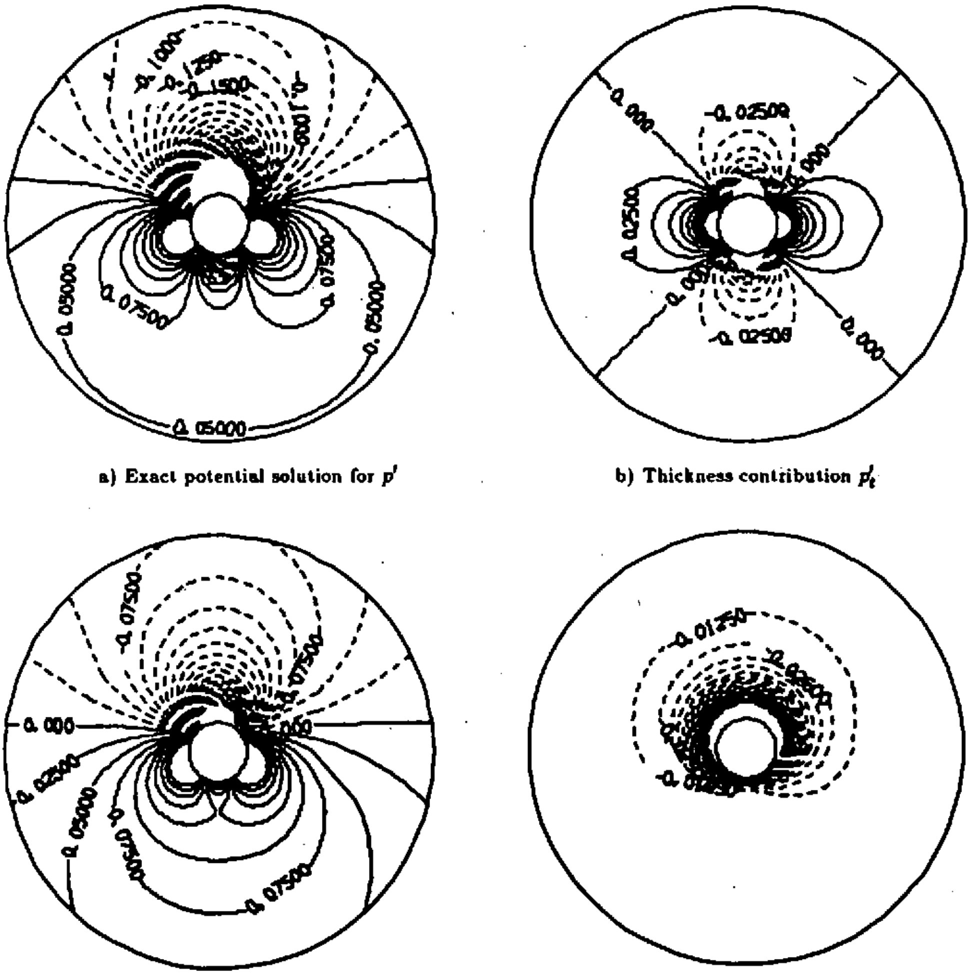

a) Exact potential solution for p'

b) Thickness contribution p'_t

c) Loading contribution p'_l

d) Quadrupole contribution p'_q

Figure 1. The perturbation pressure for a flow around a cirular cylinder, radius $a = 1.0$, with a velocity $v = 1.0$, and circulation $K = \pi$.

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This new equation is similar to Powell's theory of vortex sound */11/* where the quadrupole source region is identified with the vorticity of compact eddies in the flow. The second source term in equation (15) is restricted to the region in the flow where the vorticity is nonzero, while the third source term is written in terms of a vortex sheet of strength $u_t - v_t$ over the surface, since $\mathbf{u} - \mathbf{v} = (u_t - v_t)\hat{\mathbf{t}}$ on the surface. Equation (15) suggests writting the FW-H equation in terms of the variable $B = p' + \frac{1}{2}\rho u^2$ which is the ρ = *contant* form of the variable B Howe */12/* used for his nonlinear analogy. This variable then eliminates the volume source terms if the flow is irrotational everywhere outside of $f = 0$. Now the contribution to p' from the volume source is $-\frac{1}{2}\rho u^2$, exactly as in the case of the circular cylinder. In the following problems, it will be possible to calculate the exact potential solution and then directly identify the volume and surface contributions to the form of the FW-H equation given in equation (15).

Aerodynamic Implications- Actually the distinction between the quadrupole source of equation (5) and the volume source terms of equation (15) is an important one. If the variable B is used along with the three dimensional Green's function for the Laplace equation in unbounded space, an integral representation of equation (15) can be written

$$B - \frac{1}{4\pi}\int_{f=0} \frac{B\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}}{r^2}dS = -\frac{1}{4\pi}\int_{f=0} \frac{\rho v_n(\mathbf{u}-\mathbf{v})\cdot\hat{\mathbf{r}}}{r^2}dS - \frac{1}{4\pi}\int_{f=0} \frac{d\mathbf{v}}{dt}\cdot\hat{\mathbf{n}}\frac{\delta(f)}{r}dS \qquad (16)$$

This is a singular Fredholm integral equation of the second kind for the unknown variable B, for which the solution is desired. In */8/*, Brandão has derrived a similar equation in which the quadrupole (of equation 5) and unsteady terms were neglected. In that case, B is replaced by p' and the $\mathbf{u}\cdot\hat{\mathbf{r}}$ term is dropped from the RHS, whereas if the volume terms and unsteady term of equation (15) are negleced, the only change to equation (16) is that the variable B is reduced to p'. Without the full RHS of equation (16), Farassat and Myers*/3/* have shown that the angle of attack problem becomes an eigenvalue problem and cannot be solved, as */8/* confirms through experience. This is true because the term $\mathbf{u} - \mathbf{v}$ on the RHS of equation (16) represents the the local vorticty on the surface due to the boundary layer and any bound circulation. If the $\mathbf{u}\cdot\hat{\mathbf{r}}$ term is neglected, then no mechanism is available to generate the lift.

The Joukowski Airfoil

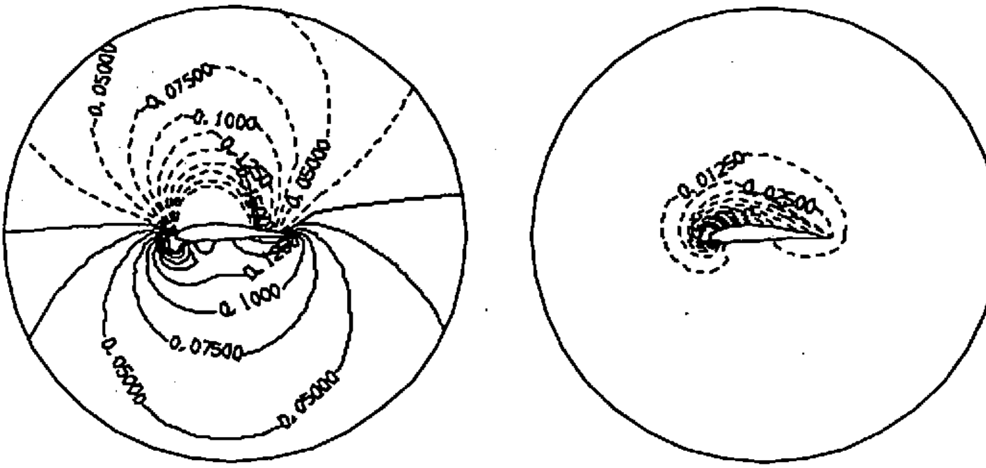
Now consider the case of a Joukowski airfoil in incompressible flow. The exact solution is readily obtained using the Joukowski transformation, $\zeta = z + 1/z$, to transform the complex velocity potential $w(z)$ for the circular cylinder. The perturbation pressure p' can be written

$$p' = -\frac{1}{2}\rho\left|\frac{dw}{d\zeta}\right|^2 + \mathrm{Re}\left(V\frac{dw}{d\zeta}\right) \qquad (17)$$

where the airfoil has a steady complex velocity $V = v(\cos\alpha + i\sin\alpha)$ and $dw/d\zeta$ is the conjugate of the complex fluid velocity. The value for $dw/d\zeta$ is most easily obtained by transforming the solution for the circular cylinder with a freestream moving past into the ζ plane and then subtract the freestream. This gives

$$\frac{dw}{d\zeta} = \left(\overline{V} - \frac{Va^2}{(z-z_o)} + \frac{iK}{2\pi(z-z_o)}\right)\frac{d\zeta}{dz} - \overline{V} \qquad (18)$$

where $\overline{V}$ is the conjugate of V and z_o is the center of the circular cylinder. From the previous discussion, it is clear that the volume source contribution of equation (15) is $-\frac{1}{2}\rho u^2 = -\frac{1}{2}\rho|dw/d\zeta|^2$



a) Surface source contribution to p' b) Volume source contribution to p'

Figure 2. The perturbation pressure components for a flow ($v = 1.0$) about a Joukowski airfoil ($a = 1.13$, $z_o = -.11 + .10i$) at $\alpha = 5$deg.

while the surface source contribution is given by $-\rho d\phi/dt = \mathrm{Re}(V dw/d\zeta)$. In Figure 2, the relative contribution of the volume and surface sources for a cambered airfoil at angle of attack are compared.

Clearly for the thin airfoil of figure 2, the volume source term is small except near the stagnation points. This observation is in fact the basis of thin airfoil theory for which p' is approximated as

$$p' \approx -\rho \frac{d\phi}{dt} \tag{19}$$

which is exactly the contribution from the surface source terms in equation (15). Therefore, neglecting the volume terms is justified by the same assumptions used in thin airfoil theory, and conversely, the volume source should not be neglected if the pressure field near a *thick* body is desired.

Circular Cylinder with a Vortex

As a final example, consider a circular cylinder moving past a vortex in the viscinity of the cylinder. In this case the complex velocity potential can be found using the Milne-Thompson circle theorem to be the sum of the velocity potential of the cylinder alone, a vortex of equal strength to the free vortex at the center of the cylinder and a vortex of equal strength and opposite sense at the image point $z_2 = a^2/\bar{z}_1$, if z_1 is the complex coordinate for the image vortex. The complex velocity potential for the problem is

$$w(z) = \frac{-Va^2}{z} + \frac{i(K+\Gamma)\ln(z)}{2\pi} + \frac{i\Gamma\ln(z-z_1)}{2\pi} - \frac{i\Gamma\ln(z-z_2)}{2\pi} \tag{20}$$

after dropping a constant. The pressure p' is then found to be

$$p' = -\frac{1}{2}\rho\left|\frac{dw}{dz}\right|^2 - \rho\mathrm{Re}\left(\frac{dw}{dt}\right) \tag{21}$$

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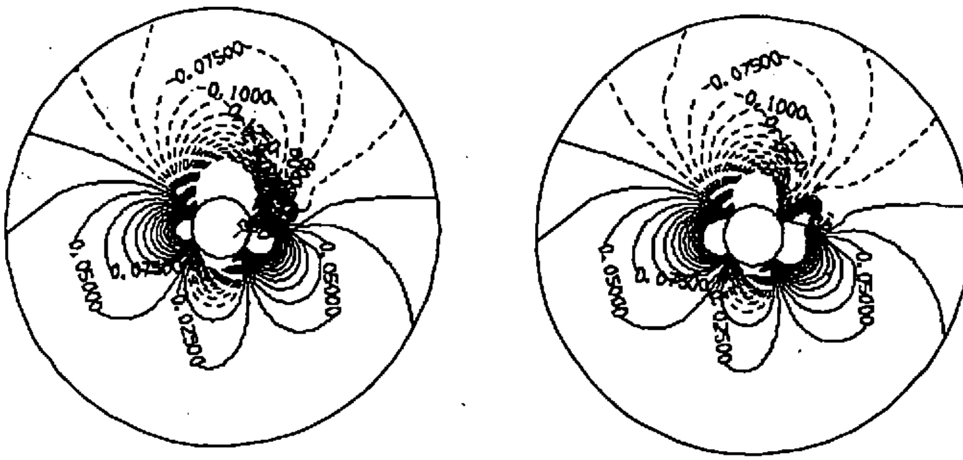

a) Exact potential solution for p' b) Approximate solution $p' \approx -\rho d\phi/dt$

Figure 3. Comparison of p' and $-\rho d\phi/dt$ for a circular cylinder moving in proximity to a free vortex. ($V = 1.0$, $K = 2.0$, $\Gamma = -2.0$, Vortex position $(r_1, \theta_1) = (2., 15 \text{ deg.})$)

where

$$\frac{dw}{dt} = V\left(-\frac{Va^2}{z^2} + \frac{i(K+\Gamma)}{2\pi z}\right) + \frac{i\Gamma}{2\pi}\left(-\frac{V_1}{(z-z_1)} + \frac{V_2}{(z-z_2)}\right) \tag{22}$$

and V_1 and V_2 are the complex velocities of the free and image vortices.

This particular problem highlights a situation where the second volume term in equation (15) must not be neglected. Since the vorticity is concentrated at the point $\mathbf{x}_1$, the vorticity vector $\boldsymbol{\zeta}$ can be written $\Gamma\delta(\mathbf{x}-\mathbf{x}_1)\hat{\mathbf{k}}$ where Γ is the strength of the vortex and $\hat{\mathbf{k}}$ is the unit vector $\hat{\mathbf{n}} \times \hat{\mathbf{t}}$. The pressure contribution due to the second term in equation (15) may then be written

$$p'_v = \frac{1}{2\pi}\nabla \cdot \int_V \Gamma\hat{\mathbf{k}} \times \mathbf{u}\delta(\mathbf{y}-\mathbf{y}_1)\ln|\mathbf{x}-\mathbf{y}|d\mathbf{y} = -\frac{\Gamma\hat{\mathbf{k}} \times \mathbf{u}|_{\mathbf{x}_1} \cdot (\mathbf{x}-\mathbf{x}_1)}{2\pi|\mathbf{x}-\mathbf{x}_1|^2} \tag{23}$$

If this result is rewritten in terms of complex variables it becomes

$$p'_v = -\frac{i\Gamma V_1}{2\pi(z-z_1)} \tag{24}$$

which is recognized immediately in equation (22). The acoustic solution can be thought of as that for the circular cylinder alone, with circulation $(K+\Gamma)$, superimposed with the solutions for the free and image vortices. A logical approximation to the volume source is to again neglect the $\frac{1}{2}\rho u^2$ part of p' and only include the effects of the free and image vortices as given by equation (24). Figure 3 shows just this approximation compared to the full exact solution. This type of approximation is not obvious directly from equation (5).

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CONCLUDING REMARKS

The aim of this paper has been to gain more understanding of the importance of the quadrupole source in the FW-H equation. Incompressible flow about both *thick* and *thin* bodies has been considered. The circular cylinder problem has shown that the thickness and loading contribution to p' is proportional to $d\phi/dt$ when the potential ϕ is written in a frame of reference fixed to the undisturbed medium. The quadrupole contribution is just $-\frac{1}{2}\rho u^2$ in this frame. This result is true generally, for inviscid, incompressible flows, if the quadrupole is reorganized in to the form of equation (13).

With the incompressible FW-H equation in the form of equation (15), the perturbation pressure solutuion, p', may be safely approximated by the surface source terms alone away from the body. Near the body and especially on the body surface, as is the case for aerodynamics, the surface sources alone in equation (15) are equivalent to thin airfoil theory. Thus the volume sources need to be included for aerodynamic calculations around thick bodies. It is important to distinguish between the neglecting the quadrupole term in equation (5) and the volume source terms in equation (15) since the vorticity needed for steady lift generation is found to be the difference between the two assumptions. This understanding of the FW-H equation as applied to incompressible aerodynamics is believed to be new.

It has also been seen in this paper, as Powell has shown previously, that the vorticity in the fluid can be considered the acoustic pressure generation mechanism. This view identifies the source regions as vortices, boundary layers, excess vorticity generating lift, and wakes, which are tangible features in the flow. As in the case of the cylinder in the viscinity of a vortex, it should be possible to model regions of vorticity in the flow separately for acoustic calculations.

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