Proceedings of the Institute of Acoustics

ACOUSTICAL IMPEDANCE CORRECTION AT THE JUNCTION OF NON-COAXIAL CYLINDRICAL DUCTS

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1. INTRODUCTION

At low frequencies only plane waves can continuously propagate along a long section of uniform duct, but at a discontinuity in the duct cross-section higher-order, evanescent acoustic waves are produced. It is common to use a linear plane-wave method for the low frequency analysis of silencer systems which are generally composed of a series of uniform ducts with area discontinuities at their joins. Whilst the plane-wave analysis is generally acceptable, there are small but noticeable errors which are caused by the neglect of the higher-order, evanescent modes which are local to the geometric discontinuities. The use of non-planar analysis throughout the silencer system is vastly more complex and expensive than plane-wave analysis and thus, for some considerable time, attention has been given to the problem of accounting for the effects of evanescent, non-planar waves in a basically plane-wave analysis.

The technique consists of finding a relationship between the plane-wave acoustic pressure and velocity variables at either side of a geometric discontinuity, which incorporates the effect of the evanescent waves. Non-planar analysis is required to determine this relationship, and it is found that the non-planar evanescent waves give rise to an effective impedance at the discontinuity. A simple algebraic expression for the effective impedance is then obtained from a curve or surface fit to the exact solution, such that a plane-wave analysis can then incorporate non-planar effects without any further recourse to non-planar analysis. The correction terms are, strictly, only valid for isolated discontinuities but, since the evanescent waves decay rapidly, they can be used at each of a series of discontinuities along a finite silencer system.

The analysis of discontinuity impedances began with the very low frequency analysis of coaxial ducts of circular cross-section, firstly for a sudden area change [1-3] and then for a bifurcated tube [4,5]. More recently the analysis for these cases has been extended up to the maximum frequency possible, that of the cut-on frequency of the first non-planar propagating mode, and the convective effects of mean flow have also been considered [6-9]. This paper continues the series by investigating the situation where the ducts at a sudden area charge are not coaxial. The complete low frequency regime is considered, but mean flow effects are ignored since they where shown to be insignificant in the coaxial case [8]. In general, mean flow effects do make significant contribution to the overall transfer matrix at a discontinuity. However, the term in the transfer matrix which is associated with the effective discontinuity impedance from non-planar waves is not significantly altered by the presence of mean flow. The results in this paper therefore find application in the analysis of engine intake and exhaust silencers throughout the normal operating range, typically of Mach number less then 0.3.

2. ANALYTICAL METHOD

The analysis basically follows that of previous work [3,8,9] and similar notation is retained in so far as possible. A geometrical discontinuity occurs at the junction of two ducts, B and C, of circular cross-section with radii b and c respectively, $(b < c)$. In general the ducts B and C are non-coaxial and the axis of duct B is taken to be distance δ from the axis of duct C, see Figure 1. Two circular cylindrical coordinate systems are employed, (ξ, β, z) for duct B and (r, θ, z) for duct C. The z-axis is in the direction of the duct axes, with its origin in the plane of the discontinuity and the sense is positive along the direction of duct C. A low frequency solution is sought, where only the plane-wave mode can continuously propagate and the higher-order modes, which are excited at the discontinuity, are rapidly attenuated.

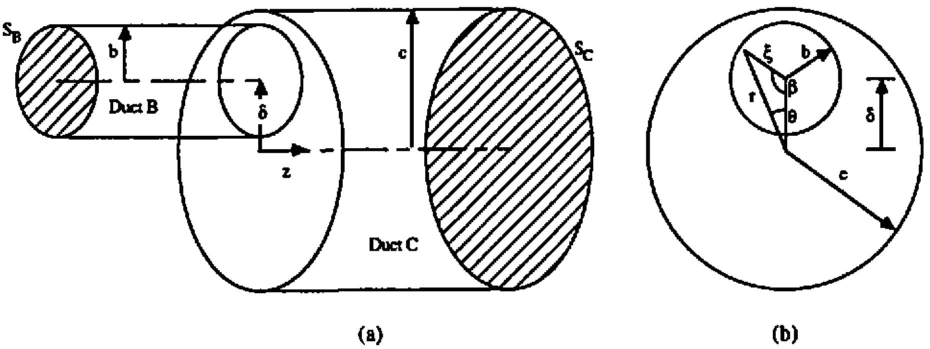

Figure 1. Geometry of the non-coaxial duct junction.

Consider harmonic waves for frequency ω in a fluid whose speed of sound is a_0, such that the linearised wave equation reduces to

$$\nabla^2 p + k^2 p = 0 \tag{1}$$

where the wavenumber $k = \omega/a_0$ and the acoustic pressure is $pe^{i\omega t}$. On the assumption that the duct walls are acoustically hard, then separation of variables solutions for each of the ducts B and C exist as follows:

$$p_B = B_0^+ e^{+ikz} + B_0^- e^{-ikz} + \sum_{\substack{m=0 \\ m \& n \neq 0}}^{\infty} \sum_{n=0}^{\infty} B_{mn} J_m\left(\frac{x_{mn}\xi}{b}\right)\cos(m\beta)e^{\gamma'_{mn}z}, \;\; z \leq 0 \tag{2}$$

$$p_C = C_0^+ e^{-ikz} + C_0^- e^{+ikz} + \sum_{\substack{m=0 \\ m \& n \neq 0}}^{\infty} \sum_{n=0}^{\infty} C_{mn} J_m\left(\frac{x_{mn}r}{c}\right)\cos(m\theta)e^{-\gamma_{mn}z}, \;\; z \geq 0 \tag{3}$$

where $\gamma'_{mn} = \left[\left(\frac{x_{mn}}{b}\right)^2 - k^2\right]^{\frac{1}{2}}$ and $\gamma_{mn} = \left[\left(\frac{x_{mn}}{c}\right)^2 - k^2\right]^{\frac{1}{2}}$. (4),(5)

The terms x_{mn} are the roots of the equations

$$\frac{d}{dx}\left[J_m(x)\right] = 0 \tag{6}$$

the lowest non-zero value of which is $x_{10} = 1.841$, see [10], for instance. The requirement that only the plane wave continuously propagates implies that γ'_{mn} and γ_{mn} are real, see equations (2) and (3), and hence, from equations (4) and (5), that $kc < 1.841$, since $b < c$. This gives the upper frequency limit at which the following analysis is generally valid.

Expressions for the particle velocity, v, in each duct follow from equations (2) and (3), since

$$\rho a_0 v_B = \frac{1}{ik}\frac{\partial p_B}{\partial z} = B_0^+ e^{ikz} - B_0^- e^{-ikz} + \sum_{\substack{m=0 \\ m\&n\neq 0}}^{\infty}\sum_{n=0}^{\infty} B_{mn} Y_{B_{mn}} J_m\left(\frac{x_{mn}\xi}{b}\right)\cos(m\beta)e^{\gamma'_{mn}z}, \tag{7}$$

$$\rho a_0 v_C = \frac{1}{ik}\frac{\partial p_C}{\partial z} = -C_0^+ e^{-ikz} + C_0^- e^{ikz} + \sum_{\substack{m=0 \\ m\&n\neq 0}}^{\infty}\sum_{n=0}^{\infty} C_{mn} Y_{C_{mn}} J_m\left(\frac{x_{mn}r}{c}\right)\cos(m\theta)e^{-\gamma_{mn}z}, \tag{8}$$

where $Y_{B_{mn}} = \frac{\gamma'_{mn}}{ik}$ and $Y_{C_{mn}} = \frac{-\gamma_{mn}}{ik}$, m and $n \neq 0$. (9)

It follows from equations (2), (3), (7) and (8) that the acoustic pressure and particle velocity in each duct at the junction $z = 0$ can be written as

$$(P_B)_{z=0} = B_{00} + \sum_{\substack{m=0 \\ m\&n\neq 0}}^{\infty}\sum_{n=0}^{\infty} B_{mn} J_m\left(\frac{x_{mn}\xi}{b}\right)\cos(m\beta) \tag{10}$$

$$(P_C)_{z=0} = C_{00} + \sum_{\substack{m=0 \\ m\&n\neq 0}}^{\infty}\sum_{n=0}^{\infty} C_{mn} J_m\left(\frac{x_{mn}r}{c}\right)\cos(m\theta) \tag{11}$$

$$(\rho a_0 v_B)_{z=0} = Y_{B_{00}} B_{00} + \sum_{\substack{m=0 \\ m\&n\neq 0}}^{\infty}\sum_{n=0}^{\infty} Y_{B_{mn}} B_{mn} J_m\left(\frac{x_{mn}\xi}{b}\right)\cos(m\beta) \tag{12}$$

$$(\rho a_0 v_C)_{z=0} = Y_{C_{00}} C_{00} + \sum_{\substack{m=0 \\ m\&n\neq 0}}^{\infty}\sum_{n=0}^{\infty} Y_{C_{mn}} C_{mn} J_m\left(\frac{x_{mn}r}{c}\right)\cos(m\theta) \tag{13}$$

The following conditions must be satisfied at the duct discontinuity:

$$\left.\begin{array}{l}(P_B)_{z=0} = (P_C)_{z=0} \\ (v_B)_{z=0} = (v_C)_{z=0}\end{array}\right\}, \qquad \text{over } S_B \tag{14}$$

$$(v_C)_{z=0} = 0, \qquad \text{over } S_C - S_B \tag{15}$$

where S_B is the region $0 \le \xi \le b$, $0 \le \beta < 2\pi$ and S_C is the region $0 \le r \le c$, $0 \le \theta < 2\pi$. It can be seen that, in general, for an exact solutions of the problem, an infinite number of evanescent waves must be considered in each duct. Conversely, if only a finite set of evanescent modes are considered, then there will be some error in the conditions given in equations (14) and (15). Denote the error in each condition as follows:

$$\left.\begin{array}{l}\epsilon_1 = (P_B)_{z=0} - (P_C)_{z=0} \\ \epsilon_2 = (v_B)_{z=0} - (v_C)_{z=0}\end{array}\right\}, \qquad \text{over } S_B \tag{16}$$

$$\epsilon_3 = (v_C)_{z=0} - 0, \qquad \text{over } S_C - S_B \tag{17}$$

One can then form the weighted residual statements

$$\int_{SB} \epsilon_1 w_q \, dS = 0 \tag{18}$$

$$\int_{SB} \epsilon_2 w_q \, dS + \int_{S_C - S_B} \epsilon_3 w_q \, dS = 0 \tag{19}$$

where the w_q are arbitrary weighting functions. Suppose we truncate m at M_b and n at N_b in duct B, giving $M_b \times N_b$ modes, and that similarly we have $M_c \times N_c$ modes in duct C, then the respective B_{mn} and C_{mn} coefficients can be found by the choice of

$$w_q = J_m\left(\frac{x_{mn}\xi}{b}\right)\cos m\beta, \quad 0 \le m \le M_b,\ 0 \le n \le N_b \text{ in equation (18)}$$

$$w_q = J_m\left(\frac{x_{mn}r}{c}\right)\cos m\theta, \quad 0 \le m \le M_c,\ 0 \le n \le N_c \text{ in equation (19)}$$

Substitution of the above weighting functions into equations (18) and (19), together with use of Graf's Addition Theorem for Bessel functions [11] which, for the geometry of Figure 1, is

$$J_m\left(\frac{x_{mn}r}{c}\right)\cos m\theta = \sum_{s=-\infty}^{+\infty} J_{m+s}\left(\frac{x_{mn}\delta}{c}\right) J_s\left(\frac{x_{mn}\xi}{c}\right)\cos(s\beta), \tag{20}$$

allows for integration by standard analytical means to give the following expressions:

$$B_{00} = C_{00} + \left[2\sum_{\substack{m=0 \\ m\&n\neq 0}}^{M_c}\sum_{n=0}^{N_c}\frac{C_{mn}}{Y_{C_{00}}C_{00}} J_m\left(\frac{x_{mn}\delta}{c}\right)\frac{J_1(x_{mn}\alpha)}{x_{mn}\alpha}\right] Y_{C_{00}}C_{00} \tag{21}$$

$$\alpha^2 Y_{B_{00}} B_{00} = Y_{C_{00}} C_{00} \tag{22}$$

$$B_{pq}\left[1-\left(\frac{p}{x_{pq}}\right)^2\right]J_p(x_{pq})=\sum_{\substack{m=0\\ m\&n\neq 0}}^{M_c}\sum_{n=0}^{N_c}C_{mn}\left[J_{m+p}\left(\frac{x_{mn}\delta}{c}\right)+(-1)^pJ_{m-p}\left(\frac{x_{mn}\delta}{c}\right)\right]$$

$$\frac{\alpha x_{pn}J_p'(x_{mn}\alpha)}{(x_{pq}^2-\alpha^2x_{mn}^2)},\quad p=0\text{ to }M_b,\quad q=0\text{ to }N_b,\quad p\&q\neq 0 \tag{23}$$

$$Y_{C_{pq}}C_{pq}\left[1-\left(\frac{p}{x_{pq}}\right)^2\right]J_p^2(x_{pq})=K_pY_{B_{00}}B_{00}J_p\left(\frac{x_{pq}\delta}{c}\right)\frac{\alpha}{x_{pq}}J_1(x_{pq}\alpha)$$

$$+2\sum_{\substack{m=0\\ m\&n\neq 0}}^{M_b}\sum_{n=0}^{N_b}Y_{B_{mn}}B_{mn}\left[J_{p+m}\left(\frac{x_{pq}\delta}{c}\right)+(-1)^mJ_{p-m}\left(\frac{x_{pq}\delta}{c}\right)\right]\frac{\alpha^3x_{pq}J_m(x_{mn})J_m'(x_{pq}\alpha)}{x_{mn}^2-\alpha^2x_{pq}^2},$$

$$p=0\text{ to }M_c,\quad q=0\text{ to }N_c,\quad p\&q\neq 0 \tag{24}$$

where $\alpha=b/c$ and $K_p=\begin{cases}2, & p=0\\ 4, & p>0\end{cases}$

3. DISCONTINUITY IMPEDANCE

A transfer matrix relationship is sought between the plane-wave acoustic pressure and volume velocity in the two ducts at the discontinuity. It follows from equations (10)-(13) that the plane-wave pressures are $p_{B_0}=B_{00}$ and $p_{C_0}=C_{00}$ and that the corresponding volume velocities are $V_{B_0}=S_BY_{B_{00}}B_{00}/\rho a_0$ and $V_{C_0}=S_CY_{C_{00}}C_{00}/\rho a_0$ for ducts B and C respectively. Hence, from equations (21) and (22), one can write

$$\begin{bmatrix}P_{B_0}\\ V_{B_0}\end{bmatrix}=\begin{bmatrix}1 & Z\\ 0 & 1\end{bmatrix}\begin{bmatrix}P_{C_0}\\ V_{C_0}\end{bmatrix} \tag{25}$$

where $Z=\dfrac{2\rho a_0}{S_C}\displaystyle\sum_{\substack{m=0\\ m\&n\neq 0}}^{M_c}\sum_{n=0}^{N_c}\frac{C_{mn}}{Y_{C_{00}}C_{00}}J_m\left(\frac{x_{mn}\delta}{c}\right)\frac{J_1(x_{mn}\alpha)}{x_{mn}\alpha},$ (26)

the effective discontinuity impedance. It transpires that Z is always imaginary, a reactance, and it is usual to write

$$Z=i\omega X. \tag{27}$$

3.1 Plane-Piston Input

The derivation of the reactance is particularly simple if one assumes that only the plane-wave is present in duct B, i.e. $B_{pq} = 0$ unless $p = q = 0$. It then follows from equations (5), (9), (22), (24), (26) and (27) that

$$X = \frac{2\rho}{\pi b\alpha} \sum_{m=0}^{M_c} \sum_{n=0}^{N_c} \frac{K_m J_m^2\left(\frac{x_{mn}\delta}{c}\right) J_1^2(x_{mn}\alpha)}{(x_{mn}^2 - m^2) J_m^2(x_{mn})[x_{mn}^2 - (kc)^2]^{\frac{1}{2}}}$$

3.2 Complete Modal Solutions.

Substitution for $Y_{B_{00}}B_{00}$ from equation (22) and for B_{pq} from equation (23) into equation (24) yields a linear system of $[(M_c + 1)(N_c + 1) - 1]$ equations in a similar number of unknowns, the coefficients $C_{pq}/(Y_{C_{00}}C_{00})$. The solution of these equations must then be substituted into equation (26) to yield the reactance, from equation (27).

4. RESULTS

For conformity with earlier work, results for the effective reactance are given in terms of the correction factor $H = (3\pi\alpha/8kc)X$. The maximum off-set distance of the axes for a given pair of ducts can be written as $\delta^* = (\delta/c)_{max} = 1 - \alpha$. Figures 2 and 3 show the variation of H with α for curves of constant $\mu\delta^*$, $\mu = 0, 0.25, 0.5, 0.75, 1.0$. In Figures 2a and 2b, results for a plane-piston input with $kc = 10^{-3}$ and $kc = 1.5$ respectively, are given. Figures 3a and 3b show similar results for the complete analysis, with non-planar evanescent waves in both ducts.

It can be seen that the effects of offset distance upon the correction factor are very significant. The frequency values of the plotted results are close to the limits of the frequency range for propagating planar waves. Thus it can be seen that, whilst frequency effects are negligible for junctions that are nearly coaxial, variations with frequency do become significant as the offset distance increases. Finally, it can be seen that the plane-piston solution gives a good approximation to the correct, full modal, solution, although the plane piston results are consistently too high. A better estimate of the true results can be obtained by simply taking 93% of the values from the plane-piston analysis. Hence the simple plane-piston analysis can be used to obtain accurate estimates of H over a very large number of data points, for varying offset distance and frequency, for use either directly as a database or in the development of empirical formulae for H.

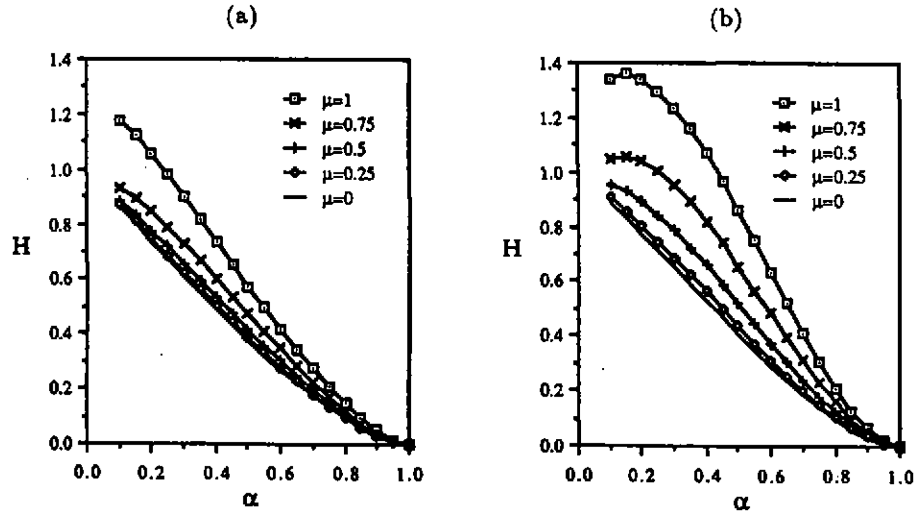

Figure 2. Correction factor for plane-piston input from duct B. (a) $k_c = 0.001$; (b) $k_c = 1.5$.

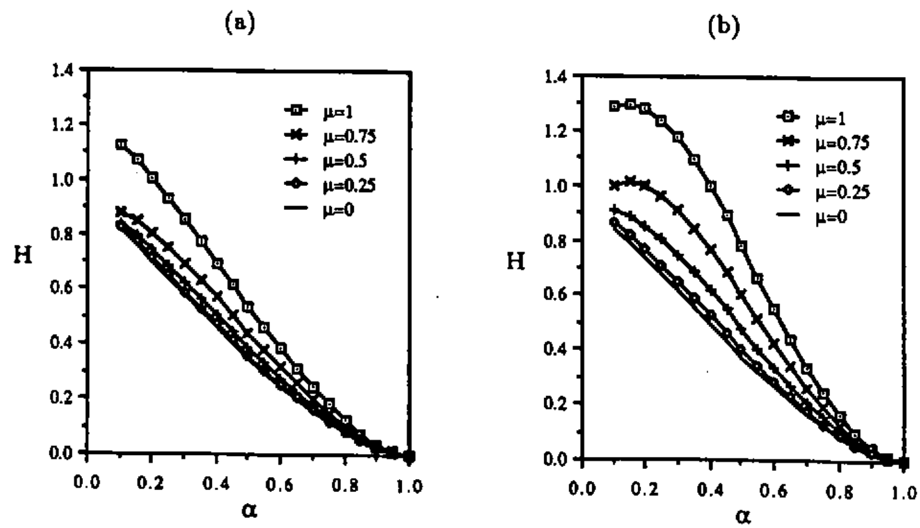

Figure 3. Correction factor for full modal analysis in ducts B and C. (a) $k_x = 0.001$; (b) $k_c = 1.5$.

5. REFERENCES

[1] J W MILES, 'The reflection of sound due to a change of cross-section in a cylindrical tube', J. Acoust. Soc. Am. 16, 14-19 (1944).
[2] J W MILES,'The analysis of plane discontinuities in cylindrical tubes. Part 1.', J. Acoust. Soc. Am. 17, 259-271 (1946).
[3] F C KARAL, 'The analogous acoustical impedance for discontinuities and constrictions of circular cross-section', J. Acoust. Soc. Am. 25, 327-334 (1953).
[4] J W MILES, 'The equivalent circuit for a bifurcated cylindrical tube', J. Acoust. Soc. Am. 19, 579-584 (1947).
[5] L L BAILIN, 'An analysis of the effect of the discontinuity in a bifurcated circular guide upon plane longitudinal waves', J. of Res. of the Nat. Bureau of Standards 47, 315-335 (1957).
[6] H HUDDE and U. LETENS, 'Scattering matrix of a discontinuity with a nonrigid wall in a lossless circular duct', J. Acoust. Soc. Am. 78, 1826-1837 (1985).
[7] J KERGOMARD and A. GARCIA, 'Simple discontinuities in acoustic waveguides at low frequencies: critical analysis and formulae', J. Sound Vib. 114, 465-479 (1987).
[8] K S PEAT, 'The acoustical impedance at discontinuities of ducts in the presence of a mean flow', J. Sound. Vib. 127, 123-132 (1988).
[9] K S PEAT, 'The acoustical impedance at the junction of an extended inlet or outlet duct', J. Sound Vib. (in press).
[10] P M MORSE, 'Vibration and Sound, 2nd ed.', McGraw-Hill, New York, (1948).
[11] G N WATSON, 'A Treatise on the Theory of Bessel Functions, 2nd ed.', Cambridge University Press, London (1958).