ERRORS IN THE LOW FREQUENCY BOUNDARY ELEMENT ANALYSIS OF CAVITY ACOUSTICS

K S Peat (1), & K A Hussain (2)

(1) Department of Mathematical Sciences, Loughborough University, Leics
(2) Motor Industry Research Association, Watling Street, Nuneaton, Warwicks

1. INTRODUCTION

The boundary element method (BEM) of analysis is, in many regards, an ideal tool for the analysis of low frequency sound fields caused by structural vibration of the bounding walls of cavities, such as in vehicle passenger compartments. As compared to the finite element method (FEM), the BEM has the potential for significant time and cost reduction at the meshing stage, where only the boundary of the cavity need be considered, and at the problem formulation and solution stage, where there is a reduction in system size. The main disadvantages is the requirement to integrate singular functions. However there is a further problem with the use of the BEM for interior acoustic analysis which has received little attention. Bernhard et al [1] demonstrated that there are inaccuracies at very low frequencies, which can be reduced by the use of very high order quadrature. Such quadrature schemes are necessary for all integrals, not just for integrals which contain singularities. The computational effort required by the high-order quadrature schemes is such that much of the advantage over the FEM is lost.

In this paper it is shown, by the use of the conventional BEM on a simple one-dimensional duct problem, that the reason for this error at low frequencies is due to ill-conditioning in the system matrix. A new BEM formulation is then given from which the first few terms of a series solution to the low frequency problem can be generated, and in this new formulation the system matrix is not ill-conditioned. The low frequency BEM formulation which is developed here is equally applicable to two- and three-dimensional problems. In addition to removing the problem of ill-conditioning, the new formulation has a further benefit in that only one matrix assembly and solution stage is needed to cover the entire low frequency region, whereas both of these steps must be repeated for each frequency value using the conventional BEM formulation.

2. CONVENTIONAL BEM FORMULATION FOR 3-D PROBLEMS

Consider a volume Ω, enclosed by a surface S, to be filled by an ideal, homogeneous, stationary fluid. For a low amplitude, harmonic disturbance of frequency ω, the sound pressure P in the cavity satisfies the Helmholtz equation

$$\partial^2P/\partial X^2 + \partial^2P/\partial Y^2 + \partial^2P/\partial Z^2 + k^2P = 0 \qquad (1)$$

where $k = \omega/a_0$ is the wavenumber and a_0 is the speed of sound. Application of the momentum equation at the surface of the cavity yields the boundary condition

$$\partial P/\partial N = -j\rho_0\omega V_n \quad \text{on } S, \qquad (2)$$

where N is the outward normal to S, ρ_0 is the mean fluid density, $j = \sqrt{-1}$ and V_n is the normal component of the particle velocity of the fluid.

Green's theorem can be used to transform equations (1) and (2) into an integral equation over the boundary S, namely [2]

$$C(\alpha)\,P(\alpha)+\frac{1}{2\pi}\iint_S P(\xi)\frac{\partial}{\partial N}\left(\frac{e^{-jkR(\alpha,\xi)}}{R(\alpha,\xi)}\right)dS=\frac{-j\rho_0\omega}{2\pi}\iint_S \frac{V_a e^{-jkR(\alpha,\xi)}}{R(\alpha,\xi)}\,dS \tag{3}$$

where α and ξ refer to two points and $R(\alpha,\xi)$ is the distance from α to ξ. The position of α is arbitrary, whilst ξ lies on S. The value of the coefficient $C(\alpha)$ depends upon whether α is inside, outside, or on the boundary.

The boundary element solution of equations (2) and (3) begins with the discretisation of the surface S into a finite number of elements [2]. The variation of P, V_a and the geometry over each element can be represented to various degrees of complexity, but in all cases, equation (3) can be reduced to the matrix form

$$[A]\,\{P\}=[B]\,\{V_a\} \tag{4}$$

where $\{P\}$ and $\{V_a\}$ are vectors of the surface pressure and normal velocity values at the nodes or control points of all the elements. The coefficients of [A] and [B] are evaluated numerically and extra care is needed when the integrals are singular. It is of particular importance to note that the wavenumber is embedded within the integrands of equation (3). Hence all of the coefficients of [A] and [B] must be evaluated separately for each frequency of disturbance considered. If the normal velocity vector $\{V_a\}$ is prescribed then one solves a system of m linear equations in m unknowns to evaluate the surface values of the pressure P. The pressure at an interior position can then be found from equation (3) with α as the interior point, since $P(\xi)$ is now known.

The formulation outlined above is not suitable at very low frequencies. For instance, Suzuki et at [3] used this formulation to determine the pressure at the centre of one end wall of a rectangular enclosed of dimensions 0.34m × 0.08m × 0.08m, when one end wall was vibrating with uniform velocity and all other walls were rigid. A mesh of 304 constant boundary elements was used, and results given over a frequency range of 250 Hz to 4 kHz showed excellent agreement with the analytical solution. However, if one uses the same mesh and formulation at low frequencies, the BEM results are found to be in serious error below 100Hz. As mentioned previously, Bernhard et al [1] have noticed that the error can be reduced by the use of very high order quadrature at low frequencies. However the integrals to be evaluated, given in equation (3), are functions only of element geometry and frequency. Thus the accuracy of the integration is greatest at low frequencies and reduces for higher frequencies, for given geometry. This implies that the accuracy requirement must increase as the frequency decreases, which in turn suggests that the conditioning of the equation system becomes poor at low frequencies. It is proved below, for a simple one-dimensional formulation, that this is indeed the case.

3. CONVENTIONAL BEM FORMULATION FOR 1-D PROBLEM

Consider the case of plane-wave sound propagation in a uniform duct of length L, for which the governing wave equation is

$$\frac{d^2p}{dx^2}+k^2\,p=0\,. \tag{5}$$

The fundamental solution to equation (5) is p = sin kr, where r = | x – xi |, xi being some chosen observation point. Thus, following the direct formulation [2], one forms the weighted residual expression

$$\int_0^L \sin kr \left(\frac{d^2p}{dx^2} + k^2 p\right) dx = 0. \tag{6}$$

Equation (6) is integrated twice by parts to give $$k\left[p\left(\frac{dr}{dx}\right)\cos kr\right]_0^L = \left[\frac{dp}{dx}\sin kr\right]_0^L \tag{7}$$

This expression is evaluated with xi = 0 and xi = L in turn, to give

$$\frac{k}{\sin kL}\begin{bmatrix} \cos kL & -1 \\ -1 & \cos kL \end{bmatrix}\begin{bmatrix} (P)_0 \\ (P)_L \end{bmatrix} = \begin{bmatrix} -(dp/dx)_0 \\ (dp/dx)_L \end{bmatrix} \tag{8}$$

where the suffixes 0 and L denote the x = 0 and x = L locations respectively. Given the boundary velocity values $(V_a)_0$ and $(V_a)_L$, then since $dp/dx = -j\rho_0 a_0\, kV_a$, equation (8) can be written as [A] {P} = {B} with known coefficients of [A] and {B} for a given problem, such that solution for {P} follows. Note, however, that

$$|\, A\,| = \frac{k^2}{\sin^2 kL}\left(\cos^2 kL - 1\right) = -k^2. \tag{9}$$

Thus, when k is small, | A | is very small and the equation system (9) becomes ill-conditioned. Small errors in the determination of the coefficients of [A] will therefore lead to large errors in the solution for {P}. The same problem occurs in matrix [A] in the three-dimensional case, equation (4).

For boundary values of $(V_a)_0 = U$ and $(V_a)_L = 0$, the exact solution of equation (8) is

$$\frac{(P)_0}{\rho_0 a_0 U} = -j \cot kL\,. \tag{10}$$

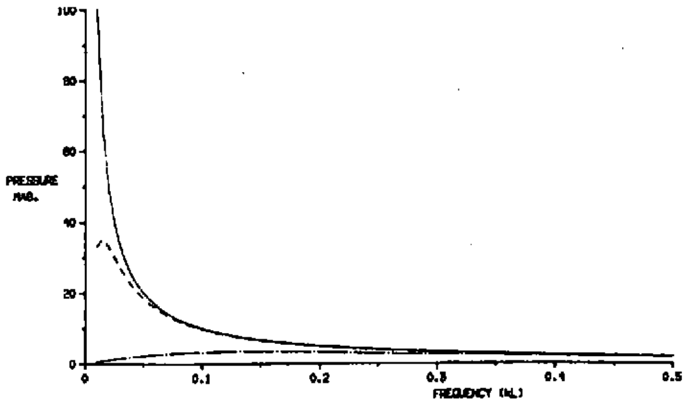

Figure 1. Pressure on the vibrating end wall of a closed duct. ______ exact solution ; __ . __ 1% error in coefficients ; _ _ _ _ 0.01% error in coefficients.

This is the same duct problem as considered by Suzuki et al [3] and the exact solution at low frequencies is shown in Figure 1. Further curves are given in Figure 1 for the pressure at the vibrating end of the duct, assuming a percentage error in the off-diagonal coefficients of matrix [A] in equation (8) of 1% and 0.01%. A similar result occurs for the same error in the diagonal coefficients or for absolute errors in the coefficients of 0.01 and 0.0001. Thus it is clearly illustrated that, at low frequencies, small errors in the numerical integration lead to large errors in results when using the conventional boundary element formulation, even for the simplest of problems.

4. LOW FREQUENCY BEM FORMULATION FOR 1-D PROBLEM

It is more convenient in this case to make the dependent variable the acoustic potential Φ, where $\Phi = P/j\,\rho_0\,\omega$, and to non-dimensionalise all variables. Then

$$\frac{d^2\Phi}{d\xi^2} + (kL)^2\,\Phi = 0 \tag{11}$$

where $\xi = x/L$, L being the duct length. Consider only low frequency problems such that $(kL) < 1$ and let $(kL) = \varepsilon$. Introduce non-dimensional forms of the particle velocity and the acoustic potential as $v = V/(\varepsilon^2 c_0)$ and $\phi = \Phi/(c_0 L)$. thus equation (11) and the associated boundary conditions become

$$\frac{d^2\phi}{d\xi^2} + \varepsilon^2 = 0 \quad , \qquad \text{with} \qquad \frac{d\phi}{dn} = -\varepsilon^2\,v_a\,, \tag{12a,b}$$

where n is the outward normal to the boundary.

Consider a series expansion of the acoustic potential of the form

$$\phi = \phi_0 + \varepsilon^2\,\phi_1 + \varepsilon^4\,\phi_2 + \ldots \tag{13}$$

Substitute this series expansion into equations (12) and equate terms of similar order in ε, to give

$$\frac{d^2\phi_0}{d\xi^2} = 0, \quad \frac{d^2\phi_1}{d\xi^2} + \phi_0 = 0, \quad \frac{d^2\phi_2}{d\xi^2} + \phi_1 = 0, \quad \frac{d^2\phi_3}{d\xi^2} + \phi_2 = 0 \ldots \qquad \text{(14a,b,c,d ...)}$$

together with the associated boundary conditions $\quad \frac{d\phi_0}{dn} = 0, \quad \frac{d\phi_1}{dn} = v_a, \quad \frac{d\phi_2}{dn} = 0$ ...(15a, b,c ...)
at both boundaries.

Clearly, the solution for ϕ_0 is simply

$$\phi_0 = \text{constant}\,. \tag{16}$$

An expression for the actual value of the constant ϕ_0 in terms only of boundary values follows from integration of equation (14b) over the domain, since

$$\left[\frac{d\phi_1}{d\xi}\right]_0^1 + \phi_0\left[\xi\right]_0^1 = 0 \tag{17}$$

Hence $$\phi_0 = -\left[\left(\frac{d\phi_1}{d\xi}\right)_1 - \left(\frac{d\phi_1}{d\xi}\right)_0\right] = -[(v_a)_1 + (v_a)_0] = -\frac{Q_{net}}{\Omega} \tag{18}$$

where Q_{net} is the net acoustic volume velocity outflow to the duct and Ω is the duct volume. The next step is to solve the Poisson equation (14b) by the boundary element method. The fundamental solution of Laplace's equation, namely $r = |\xi - \xi_i|$, is used to form the weighted residual statement

$$\int_0^1 r\frac{d^2\phi_1}{d\xi^2}\,d\xi + \phi_0\int_0^1 r\,d\xi = 0\,. \tag{19}$$

This equation is integrated twice by parts to give

$$\left[\frac{dr}{d\xi}\phi_1\right]_0^1 = \left[r\frac{d\phi_1}{d\xi}\right]_0^1 + \phi_0\left[r\xi - \frac{dr}{d\xi}\frac{\xi^2}{2}\right]_0^1 \tag{20}$$

and is evaluated with $\xi_i = 1$ and $\xi_i = 0$ in turn, to give

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (\phi_1)_0 \\ (\phi_1)_1 \end{bmatrix} = \begin{bmatrix} \phi_0/2 + (v_a)_0 \\ \phi_0/2 + (v_a)_1 \end{bmatrix}. \tag{21}$$

Equation (20) makes it clear that this is a redundant set of equations, thus one can solve for the ϕ_1 values only to within some arbitrary constant, say $\bar{\phi}_1$. Let

$$\phi_1 = \phi_1' + \bar{\phi}_1\,, \qquad (\phi_1')_1 = 0. \tag{22a,b}$$

Then $$(\phi_1')_0 = \phi_0/2 + (v_a)_0\,. \tag{23}$$

The constant $\bar{\phi}_1$ can be found from the integration of equation (14c) over the domain, since

$$\left[\frac{d\phi_2}{d\xi}\right]_0^1 + \left[\phi_1\,\xi\right]_0^1 - \int_0^1 \frac{d\phi_1}{d\xi}\xi\,d\xi = 0, \tag{24}$$

and therefore, using equations (14b) and (15b),

$$[\phi_1 \xi]_0^1 - \left[\frac{d\phi_1}{d\xi}\frac{\xi^2}{2}\right]_0^1 - \phi_0 \int_0^1 \frac{\xi^2}{2}\, d\xi = 0. \tag{25}$$

Thus

$$\bar{\phi}_1 [\xi]_0^1 = \left[\frac{d\phi_1}{d\xi}\frac{\xi^2}{2}\right]_0^1 - [\phi_1' \xi]_0^1 + \phi_0 \left[\frac{\xi^3}{6}\right]_0^1 \tag{26}$$

and the right-hand side contains expressions on the boundary which are either known, or can be calculated. Hence

$$\bar{\phi}_1 = (v_a)_1 + \phi_0/6 \tag{27}$$

Higher-order terms of the series can be found in a similar manner, with the simplification that the v_a terms are now zero. Thus the weighted residual form of equation (14c) is integrated several times by parts, with the use of equation (14b), to give

$$\left[r\frac{d\phi_2}{d\xi}\right]_0^1 - \left[\frac{dr}{d\xi}\phi_2\right]_0^1 + [r\phi_1 \xi]_0^1 - \left[\frac{d}{d\xi}(r\phi_1)\frac{\xi^2}{2}\right]_0^1 + \left[\frac{dr}{d\xi}\frac{d\phi_1}{d\xi}\frac{\xi^3}{3}\right]_0^1 + \phi_0\left\{\left[\frac{dr}{d\xi}\frac{\xi^4}{8}\right]_0^1 - \left[r\frac{\xi^3}{6}\right]_0^1\right\} = 0. \tag{28}$$

This expression is evaluated with $\xi_i = 1$ and $\xi_i = 0$ in turn, to give

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (\phi_2)_0 \\ (\phi_2)_1 \end{bmatrix} = \begin{bmatrix} -(v_a)_1/3 - \phi_0/8 + (\phi_1)_1/2 \\ (v_a)_1/3 - \phi_0/8 - (\phi_1)_1/2 \end{bmatrix}. \tag{29}$$

Once again, one can solve for ϕ_2 only to within some arbitrary constant, say $\bar{\phi}_2$. With $\phi_2 = \phi_2' + \bar{\phi}_2$, and $(\phi_2')_1 = 0$. Then

$$(\phi_2')_0 = (v_a)_1/3 - \phi_0/8 + \bar{\phi}_1/2. \tag{30}$$

The constant value $\bar{\phi}_2$ from the integration of equation (14d) over the domain, with the use of equations (14b), (14c), (15b) and (15c), which gives

$$\bar{\phi}_2 = \phi_1/6 - (v_a)_1/24 - \phi_0/120. \tag{31}$$

One can evaluate further higher-order terms in a similar fashion. Note in particular that the matrix [A] in equations (21) and (29) is identical. Thus when considering higher-order terms it is only necessary to evaluate the right-hand sides and to manipulate the right-hand sides for equation solutions. This is equally true, and particularly relevant, to solutions of two- or three-dimensional problems, although in these cases it is necessary to integrate known functions over the boundary of the domain in order to determine the right-hand sides.

Application of the above formulation to the same duct problem as considered in Section 3 gives, from equations (13), (18), (22), (23), (27), (30) and (31)

$$(\phi)_0 = (-u)\left(1 - \frac{\varepsilon^2}{3} - \frac{\varepsilon^4}{45} + \ldots\right), \qquad \text{or} \qquad \frac{(P)_0}{\rho_0 a_0 U} = -j\left(\frac{1}{\varepsilon} - \frac{\varepsilon}{3} - \frac{\varepsilon^3}{45} + \ldots\right). \tag{32a,b}$$

Figure 2 illustrates the form of solution for an increasing number of terms of the series. It is seen that the low frequency solution requires very few terms of the series expansion, thus the solution technique is practical. Furthermore, an error of even 1% in the coefficients of matrix [A] now has a negligible effect upon the solution.

It is seen from equations (10) and (32) that the analytical series solution for ϕ in this case is in fact

$$\frac{\phi}{(-u)} = \varepsilon \cot \varepsilon = \frac{\varepsilon \cos \varepsilon}{\varepsilon(1-\beta)} = \left(1 - \varepsilon^2/2! + \varepsilon^4/4! \cdots\right)\left(1 + \beta + \beta^2 + \cdots\right), \; |\beta| < 1, \qquad (33)$$

where $\beta = \varepsilon^2/3! - \varepsilon^4/5! + \cdots$.

Thus $$\frac{\phi}{(-u)} = 1 - \frac{\varepsilon^2}{3} - \frac{\varepsilon^4}{45} + \frac{2}{945}\varepsilon^6 \cdots \qquad (34)$$

and is convergent for $\left|1 - \frac{\sin\varepsilon}{\varepsilon}\right| < 1$, or $0 < \frac{\sin\varepsilon}{\varepsilon} < 2$, hence $\varepsilon < \pi$.

The truncated series can only be expected to give an accurate solution for $\varepsilon < 1$ but, since the coefficients in equation (34) are seen to decrease quickly in magnitude, the range of accuracy is greater than this, as seen in Figure 2.

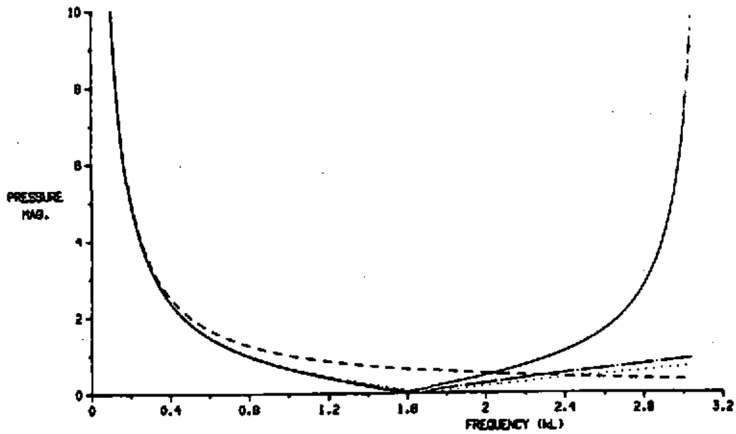

Figure 2. Pressure on the vibrating end wall of a closed duct of length L ______ exact solution ; - - - - - 1 term, ... 2 terms, __ . __ 3 terms of series solution.

5. CONCLUSIONS

At very low frequencies, errors which arise from use of the conventional BEM are due to ill-conditioning in the system matrix. It is possible to remove the ill-conditioning problem by the use of a series solution BEM formulation. This method requires only one analysis for the entire low frequency region and thus gives significant time saving as compared to the conventional BEM formulation, in addition to the gains in accuracy. This paper has concentrated on simple one-dimensional problems in order to highlight the causes of the errors and the basis of the new technique, both of which are unchanged for two- and three-dimensional problems [4], when the relevance of the method becomes apparent.

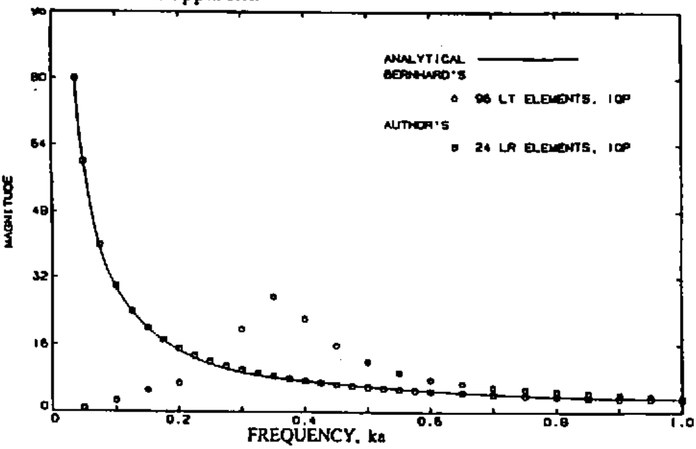

Figure 3. Pressure at the mid-radius position of a vibrating sphere of radius a.

An example of the use of the method for a three-dimensional problem is shown in Figure 3, which gives the solution for the vibrating sphere problem considered by Bernhard et at [1]. It is seen that the new formulation has significant accuracy gains over the conventional formulation even when the quadrature scheme is single-point.

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