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FINITE ELEMENT FORMULATIONS AND SOME ADAPTATIONS FROM THE ANALYSIS OF SOLID CONTINUA

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INTRODUCTION: Over the last decade, the matrix displacement method, which is known to be a finite element analogue of the Rayleigh-Ritz method, has been widely used for vibration analysis of structural components. It is possible to develop new and equally useful finite element methods adapting other known approximate methods in the analysis of solid continua. In this paper, we present two methods, one based on the modified Rayleigh-Ritz method following Reissner and the other based on the Galerkin method. An assessment of these methods in comparison with the conventional displacement method is given with the aid of simple examples.

NOMENCLATURE:

$[a]$	: displacement transformation matrix
$[b_I]$	: load transformation matrix
$\{I_z\}$	: inertia loading
m	: mass per unit length
v_i	: displacement distribution over i^{th} element
$\{z\}$	: local or element degrees of freedom
$\{Z\}$	: global or structural degrees of freedom
ξ	: local coordinate in the element
ω	: natural frequency

THE MODIFIED RAYLEIGH-RITZ METHOD:

Basis: For the sake of brevity and clarity we describe the method with reference to vibrations of beams. In this method one starts with assumed displacements over the structure. Assuming sinusoidal vibration, the inertia loading in the structure is the product of the square of the frequencies, mass per unit length of the beam and the local displacement. Minimization of a function, defined as the difference of the strain energy corresponding to the inertia loading and the kinetic energy corresponding to assumed displacement distribution, provides the procedure for approximating the natural frequencies.

Finite Element Analogue: The principal steps in this procedure are as follows:

(1) Divide the structure into a set of sub-domains called 'elements'.

(2) Assume a suitable displacement distribution over each element. In the case of a beam we choose a linear displacement distribution as

$$V(\xi_1) = [\xi_1 \;\; 1 - \xi_1]\{z_1 \; , \; z_2\} = [f(\xi_i)]\{z_i\} \qquad ..(1)$$

(3) Derive the mass matrix of the element using the principle that the derivative of the kinetic energy with respect to z_1 gives the inertia load in that direction. With Eq.(1), the mass matrix is easily seen to be

$$[\bar{m}] = \frac{m}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad ..(2)$$

(4) Total kinetic energy of the structure is

$$T = \frac{1}{2}\,\omega^2\,\{Z_i\}^T\,[a]^T\,[\bar{m}]\,[a]\,\{Z_i\} \qquad ..(3)$$

(5) The strain energy corresponding to the inertia loading for the assumed displacements (Eq.1) is

$$U = \frac{1}{2}\,\{I_Z\}^T\,[b_I]^T\,[f_1]\,[b_I]\,\{I_Z\} \qquad ..(4)$$

where,

$\{I_Z\} = \omega^2\,[f(\xi_i)]\,[a]\,\{Z_i\}$

$[b_I]$ = load transformation matrix, relating $\{I_Z\}$ with assumed generalized loads in the element.

It may be noted here, that U of Eq.(4), corresponds to the strain energy associated with the deformed shape of the structure under the inertia loading and it is expressed as in Eq.(4) for convenience.

(6) The condition that $\delta(U-T) = 0$ gives the necessary equations to complete the solution.

THE GALERKIN METHOD:

Basis: In this method one starts with the displacement expressed in terms of a set of displacement functions each satisfying the boundary conditions. This expression substituted in the governing differential equations, yields an error function. Integrals of the error function multiplied by each component function of the assumed expression for displacement distribution, are set to zero, to obtain the necessary equations for completing the solution

Finite Element Analogue: We will describe the procedure here with reference to flexural oscillations of uniform beams.

(1) Divide the structure into a set of sub-domains called 'elements'.

(2) Assume a suitable displacement distribution in each element. For a beam the displacement distribution over the element, may be taken as

$$V_i\,(\xi_i) = [f(\xi_i)]\,\{z_i\} \qquad ..(5)$$

(3) Substitute Eq.(5) into the governing differential equation to get the error function. For free flexural oscillations of beams the governing equation is $V^{iv} + \lambda^2 V = 0$. In this case the error function is

$$\epsilon_i(\xi_i) = [f^{iv}(\xi_i) + \lambda^2 f(\xi_i)]\,\{z_i\} \qquad ..(6)$$

(4) Assumed displacement distribution over the entire structure in terms of the global coordinates $\{Z\}$ is easily seen to be

$$Z = [f_i(\xi_i)]\,[a]\,\{Z_i\} \qquad ..(7)$$

and the corresponding error function becomes

$$\epsilon = [f_i^{iv}(\xi_i)]\,[a]\,\{Z_i\} + \lambda^2 [f_i(\xi_i)]\,[a]\{Z_i\} \qquad ..(8)$$

so that, following the Galerkin procedure, one gets the final equation as

$$[a]^T [e_k]\,[a] + \lambda^2 [a]^T [e_m]\,[a] = 0 \qquad ..(9)$$

where,

$$[e_{k_i}] = \int_0^1 [f]^T\,[f^{iv}]\,d\xi_i \;\;\&\;\; [e_{m_i}] = \int_0^1 [f]^T [f]\,d\xi_i \qquad ..(10)$$

We may note here that, if the rod is of variable cross-section, Eqs.(10) get modified. Using Eq.(9) eigenvalues and eigenvectors can be computed.

ILLUSTRATIONS: The first three natural frequencies of a simply supported uniform beam are estimated using various methods and the errors are compared in Table 1. The relative superiority of these methods is clear from these comparisons.

Table 1 : Percentage Error in Eigenvalues for a Simply Supported Beam

Order of the matrix and No. of degrees of freedom	Conventi-onal Matrix Displace-ment Method	Finite Element Method based on Modified Rayleigh-Ritz Method	Finite Element Method based on Galerkin Method	Number of Mode
4	0.791	0.0610	0.00005	First
8	0.0517	0.0034	0.00002	
4	23.1	1.4521	0.00600	Second
8	0.791	0.0610	0.00005	
4	53.74	12.0609	5.67572	Third
8	3.0	0.3668	0.00396	

CONCLUDING REMARKS: We have presented here, two finite element methods, based on the modified Rayleigh-Ritz and the Galerkin methods, for the analysis of natural vibrations of structural components. For the example analysed these methods are found to be superior to the conventional matrix displacement method for estimating eigenvalues. It is worth exploring the possibility of developing useful finite element analogues of other powerful approximate methods which are available for the analysis of solid continua.