THE EFFECT OF COORDINATE TRANSFORMATION FUNCTION ON SCATTERING CHARACTERISTICS OF LAYERED CYLINDRICAL PENTAMODE ACOUSTIC CLOAK

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As a newly-developd method, acoustic cloak made of pentamode materials is on its speedway to the promising potential application. For the physical realization of a pentamode cloak, some forms of discretization were utilized to the ideal continuous parameters obtained from the transformation method, which in the expense of inherent acoustic scatterings. It is necessary to investigate a better way to design a practical cloak with good performance. The present work concerns the effects of coordinate transformation function on pentamode acoustic cloak with a quantity of discrete homogeneous layers. By using the polynomial coordinate transformations with a free transformation order number, the scattering cross sections (TSCS) of cloaks designed by different coordinate transformation function are discussed. It has been found that a proper strategy for the choice of transformation function can help to optimize the field distribution inside the cloak and the invisible performance of the cloak could be improved with minimization of SCS. Rigorous Scattering characteristics analyses are presented to give a physical insight of it. This paper offers a useful reference for future fabrication of realistic acoustic pentamode cloak with only a few layers.

1. Introduction

In recent years increasing attention has been focused on cloaking a target from detective wave[1-8], which can render arbitrarily sized and shaped objects invisible. Material design for such cloak is only possible recently based on transformation method, initially advanced for electromagnetic waves[1, 2] and later extended to acoustic waves[3-8]. Acoustic cloak was first explored by recognizing similarity between acoustic and electromagnetic wave equations[4, 5] and have to resort to metafluids with isotropic stiffness but anisotropic density, different design schemes making use of the acoustic layered microstructure and metamaterials[9−11] are suggested to realize acoustic cloak physically. These pioneering designs are classified as inertial cloak(IC) after Norris[12].

Norris proved that under a space mapping(Fig. 1(a)), transformed acoustic wave equation with PM material expressed in pseudo pressure is form invariant. The acoustic cloak can invoke PM materials with both anisotropic stiffness and anisotropic density, covering the IC as a special case. And another special case, PM cloaks, is discussed. Which with anisotropic stiffness but isotropic density. Over classical inertial cloaks, PM cloaks avoid mass singularity and can be engineered with pure solid materials. Since then much progress has been made for PM cloaks. Gokhale et al. [13] discussed choice of transformation function to get different material distribution. Scandrett et al. [14] studied layered approximation of PM cloaks. Fuelled by transformation acoustics based on PM materials, investigation on PM materials themselves also becomes active recently[15].

As a newly-developed method, acoustic cloak made of pentamode materials is on its speedway to the promising potential application. However, physical fabrication of pentamode cloak with continuously varying material parameters can be a tough work. Therefore some forms of discretization and truncation were used to facilitate the physical realization, which make the cloak imperfect and lead to intrinsic scatterings. It is important to concern the layering effects of pentamode acoustic cloak composed of discontinuous layers of homogeneous anisotropic materials. And according to coordinate transformation theory, cloaks are obtained by coordinate transformation compresses space from the virtual space into transformed space (physical space). There are an infinite number of transformations that can perform the compression, which provides an important design freedom for cloak designing. So far different coordinate transformation functions have been adopted for designing acoustic cloaks with special properties such asconstant density, constant radial stiffness, constant tangential stiffness, and so on. It is of great interest to study the effect of transformation functions on the cloak's invisibility performances.

In this paper, we focus on the effect of coordinate transformation function on the cloaking performance of pentamode multilayered cloaks. By using coordinate transformations with different transform function, it has been found that a proper choice for the transformation function can help to minimize the scattering fields and yield better cloaking performance. Rigorous acoustic scattering analyses are presented to give a physical insight of it. These results offer a useful reference for future fabrication of realistic multilayered acoustic pentamode cloak.

2. Principle description

According to Norris's theory, the coordinate transform function from virtual acoustic space (r′, θ′, z′) to physical space(r, θ, z) takes the form of

$$\rho = \rho_0 f' \left(\frac{f}{r} \right)^{d-1}, \qquad K_r = \frac{K_0}{f'} \left(\frac{f}{r} \right)^{d-1}, \qquad K_t = K_0 f' \left(\frac{f}{r} \right)^{d-3} \tag{1}$$

Where f is the coordinate transform equation, d is the dimensionality of the transformation model, $d=2$ for two-dimensional model and $d=3$ for three-dimensional model. ρ and K are the density and bulk modulus tensor for the cloak, ρ_0 and K_0 are the density and bulk modulus of the host medium respectively.

The theory above permits considerable freedom in choosing the transformation f from physical to virtual space. For cylindrical cloak, transformations are derived which result in acoustic cloaks with special properties such as Eq. (2) for constant density, Eq. (3) for constant radial stiffness, Eq. (4) for power-law density, Eq. (5) for power-law radial stiffness respectively.

$$\rho = \alpha \rho_0, \qquad K_r = K_0 \frac{1}{\alpha} \left(\frac{\alpha r^2 + (1-\alpha) b^2}{r^2} \right), \qquad K_t = \frac{K_0^2}{K_r} \tag{2}$$

$$\rho = \rho_0 \frac{K_0}{K_r} \left(\frac{r}{b} \right)^{2\left(\frac{K_0}{K_r} - 1 \right)}, \qquad K_r = K_0 \frac{\ln\left(\frac{a}{b} \right)}{\ln\left(\frac{\delta}{b} \right)}, \qquad K_t = \frac{K_0^2}{K_r} \tag{3}$$

$$\rho(r) = \rho_a \left(\frac{r}{a} \right)^{\alpha}, \qquad K_r = \frac{\rho_0}{\rho_a} \frac{a^{\alpha} f^2}{r^{\alpha+2}} K_0, \qquad K_t = \frac{K_0^2}{K_r} \tag{4}$$

$$\rho = \rho_0 \frac{f^2}{r} \frac{K_0}{K_a} \frac{a^{\alpha}}{r^{\alpha+1}}, \qquad K_r = K_a \left(\frac{r}{a} \right)^{\alpha}, \qquad K_t = \frac{K_0^2}{K_r} \tag{5}$$

The parameter α can be a rational number other than an integer, which facilitate to choice different coordinate transformation that compresses the cylindrical region $0 < r' < b$ into the annular region $a < r < b$.
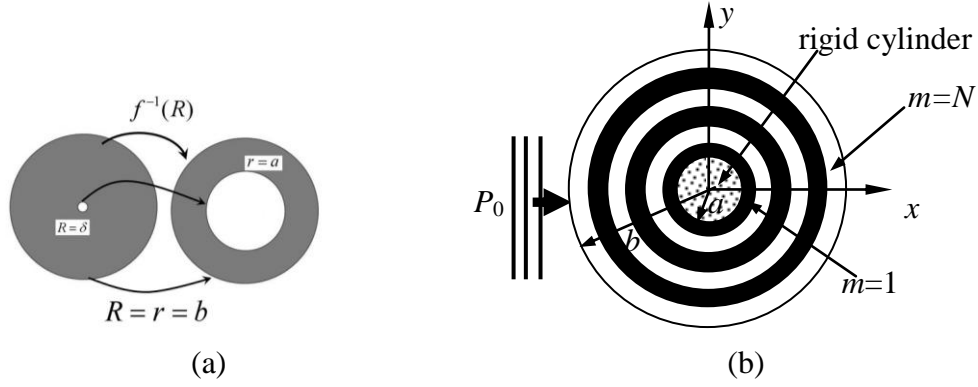
Figure 1:Illustration of a multilayered cylindrical cloak.

It is interesting to study the cloaking performance of the above cloaks. Here a multilayered cloak is used, the continuous radius-dependent, anisotropic medium could be represented approximately by N discrete layers of homogeneous anisotropic medium(Fig. 1(b)). Consider the acoustic wave scattering for an infinite rigid cylinder shelled with a concentric layered structure, as shown in Fig. 1(a). A plane wave $p_{inc} = P_0 e^{i(k_0 r \cos\theta - \omega t)}$ is assumed to impinge along the x direction upon the shelled cylinder. The pressure fields in each region can be expanded in cylindrical coordinates as :

$$p_0 = P_0 \sum_{n=-\infty}^{\infty} \left[A_{0n} J_n(k_0 r) + B_{0n} H_n^{(1)}(k_0 r) \right] e^{in\theta} \qquad (r > b)$$

$$p_{rn} = P_0 \sum_{n=-\infty}^{\infty} \left[A_{mn} J_n(k_m r) + B_{mn} H_n^{(1)}(k_m r) \right] e^{in\theta} \qquad (R_m > r > R_{m-1}) \cdot \qquad (6)$$

$$p_{in} = P_0 \sum_{n=-\infty}^{\infty} \left[A_{in\,n} J_n(k_{in} r) \right] e^{in\theta} \qquad (r < a)$$

Where J_n and H_n are the Bessel and Hankel functions of the first kind. k_0, k_m and k_{in} are the wave numbers in the corresponding layers, the suffix m indicate the cloak layer. A time dependence of the form $e^{j\omega t}$ is assumed for the acoustic field quantities but is suppressed throughout. B_{0n}, A_{mn}, B_{mn}, and A_{in} are the unknown expansion coefficients, which can be solved by matching the boundary continuity condition at $r = a$, $r = b$, and at the interfaces between each pentamode layer. The boundary continuity condition of pentamode materials are discussed in Ref[16], where Fourier expansion plays a key role.

Based on the cylindrical scattering model, the far-field total scattering efficiency or the scattering cross section (TSCS) normalized by the geometrical cross section for the multilayered cylindrical cloak is obtained as

$$Q_{sca} = \frac{2}{k_0 b} \sum_{n=-\infty}^{\infty} |b_{0n}|^2 . \qquad (5)$$

which is defined as the ratio of the total scattered power to the incident power and can be used to quantify the cloaking performance of the cloak.

3. Numerical results and discussion

To compare the cloaking performance of the pentamode cloaks with different transformation functions, Fig. 2 depicts the TSCS as a function of normalized frequency $k_0 a$. Cloaks with different transformation function, as well as uncloaked rigid cylinder are involved. In all examples, we fix $a = 0.6$m and $b = 2.0$m, and the multilayered model with 40 layers. The value of α is shown in table1.

Table 1: the value of α for the pentamode cloak.

	Transform function	α
constant density	$\rho = \alpha\rho_0,\ K_r = K_0\dfrac{1}{\alpha}\left(\dfrac{\alpha r^2 + (1-\alpha)b^2}{r^2}\right),\ K_t = \dfrac{K_0^2}{K_r}$	1.1
constant radial stiffness	$\rho = \rho_0\dfrac{K_0}{K_r}\left(\dfrac{r}{b}\right)^{2\left(\frac{K_0}{K_r}-1\right)},\ K_r = K_0\dfrac{\ln\left(a/b\right)}{\ln\left(\delta/b\right)},\ K_t = \dfrac{K_0^2}{K_r}$	0.16
power-law density	$\rho(r) = \rho_a\left(\dfrac{r}{a}\right)^{\alpha},\ K_r = \dfrac{\rho_0}{\rho_a}\dfrac{a^\alpha f^2}{r^{\alpha+2}}K_0,\ K_t = \dfrac{K_0^2}{K_r}$	-5
power-law radial stiffness	$\rho = \rho_0\dfrac{f^2}{r}\dfrac{K_0}{K_a}\dfrac{a^\alpha}{r^{\alpha+1}},\ K_r = K_a\left(\dfrac{r}{a}\right)^{\alpha},\ K_t = \dfrac{K_0^2}{K_r}$	4

The cloaking performance is evident in all the representations since the TSCS of the rigid cylinder are significantly larger than that with cloak in the entire frequency region under investigation. The triangle, circle, and solid lines in Fig. 2 show that the cloaking performances are different as different transform function is used. Comparison with power-law density, the cloaking performance becomes more evident as the transform function of constant density are used. And the transform function of power-law radial stiffness results in much reduced scattering from the cloaking structure. However the cloaking performance becomes even better for the case of constant radial stiffness.
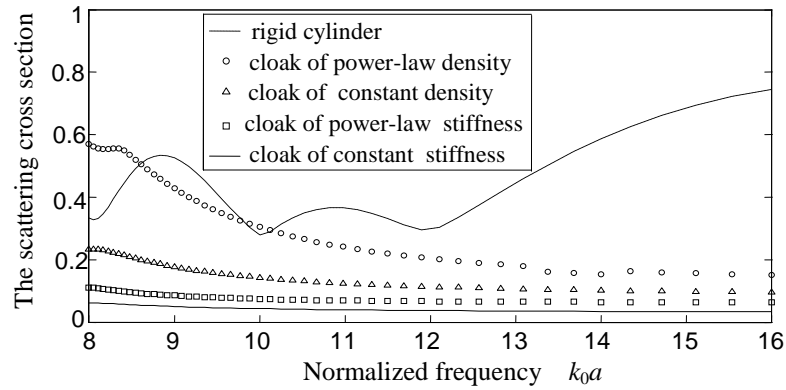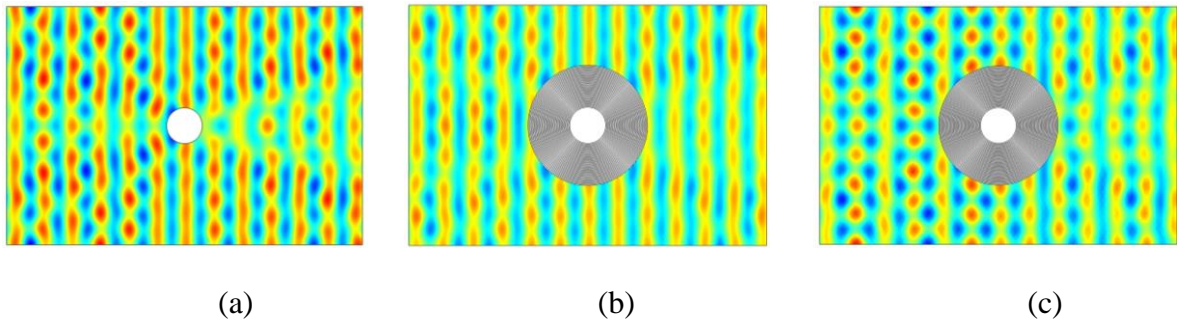

Figure 2: The scattering cross section of the pentamode cloaks with different transformation function

To clarify the variation of TSCS, we examine the acoustic pressure field distributions for different cloak shells illuminated by a normally incident plane wave, just as shown in Figures 3. Compared with the case of power-law density and constant density, the transform of constant radial stiffness results in much reduced scattering from the cloaking structure. The results are consistent with that of TSCS.
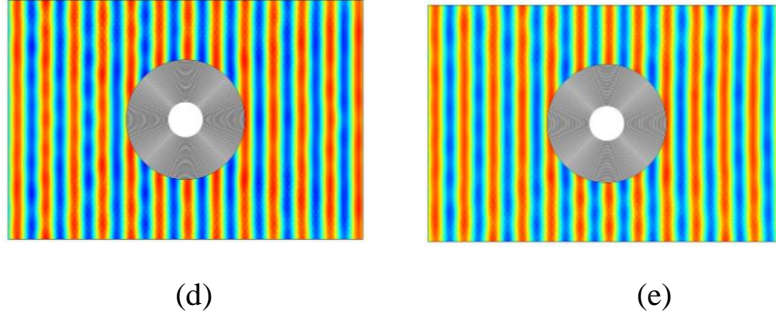

(a)　　　　　　　　(b)　　　　　　　　(c)

(d)   (e)

Figure 3: The acoustic field of the different pentamode cloak (ka=13，f=1600Hz); (a) rigid cylinder; (b) the density equation is power equation; (c) the density equation is constant; (d) the modulus equation is power equation; (e) the modulus equation is constant

To discuss the effect of coordinate transformation, we plot the propagation tracing of acoustic wave in the cloak shell with different transformation functions in Fig. 4. It can be found the rays of acoustic wave propagation in cloak shell move inward in the case of constant density and move outward in the case of constant modulus. It means the waves are bent to concentrate near the inner boundary for the former and are bent to concentrate near the outer boundary for the latter. By Eq. (1), the impedance at the cloak's outer boundary ($r = b$) matches with the surrounding medium ($\rho \cdot K_r = \rho_0 \cdot K_0$), the incident wave is reflectionless at it. Therefore the scattering wave is derived from the inner boundary of the cloak shell, where truncation is utilized to approximate the infinite values in coordinate transformation theory.And the interfaces between the adjacent discrete aniso-tropic layers inside the shell. The above discussion means the inner boundary is dominant for the scattering wave of the cloak shell. Therefore decreasing the wave near the inner boundary can de-crease the scattering.

(a)   (b)

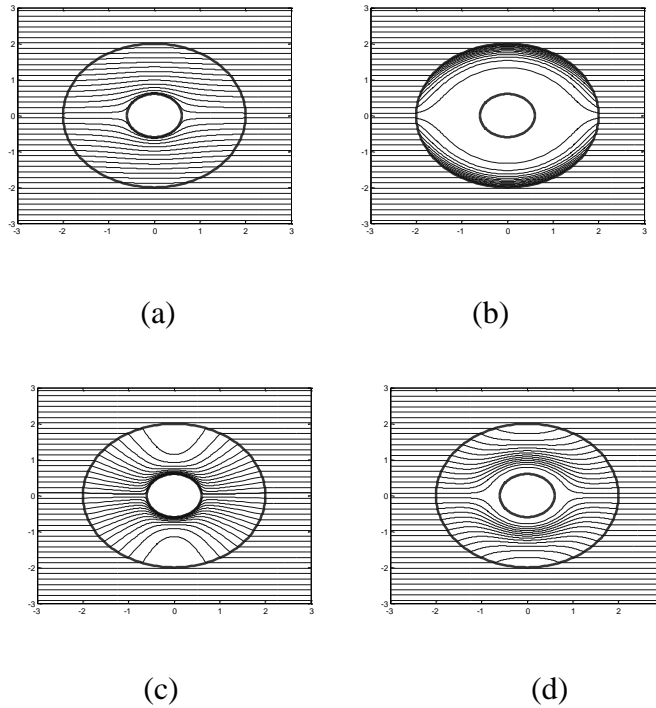(c)   (d)

Figure 4: The propagation tracing of acoustic wave in pentamode cloak; (a) the density equation is constant (b) the modulus equation is constant (c) the density equation is power equation (d) the modulus equation is power equation

To check the analyse above, we fix the transform function to Eq.(4) for power-law density and change the parameter α. The TSCS of α= -5, α=-0 and α=5 are shown in Fig. 5. It can be found the cloaking performance becomes more evident as α change from -5 to 5. The propagation tracing and field distribution of the same cloak are shown in Fig. 6 and Fig. 7. It can be found that during the course of α increasing from -5 to 5, less wave are bent to concentrate near the inner face and the cloak reduce more scattering wave. Therefore by varying the parameter α, the distribution of acoustic wave in the cloak can be changed, and a good cloaking performance can be obtained for different transform function.
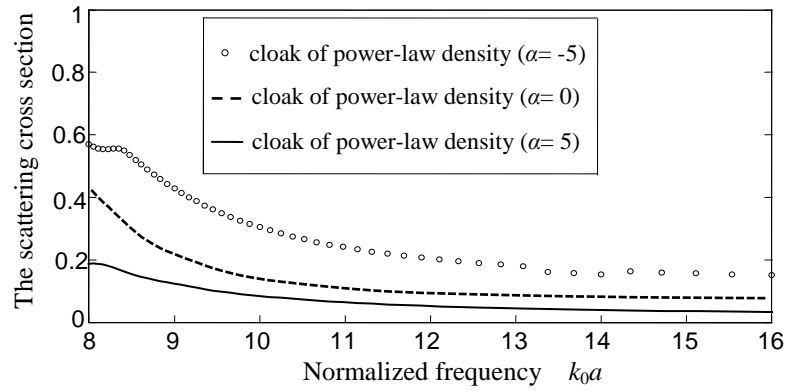
Figure 5: The scattering cross section of the pentamode cloaks with different transformation function
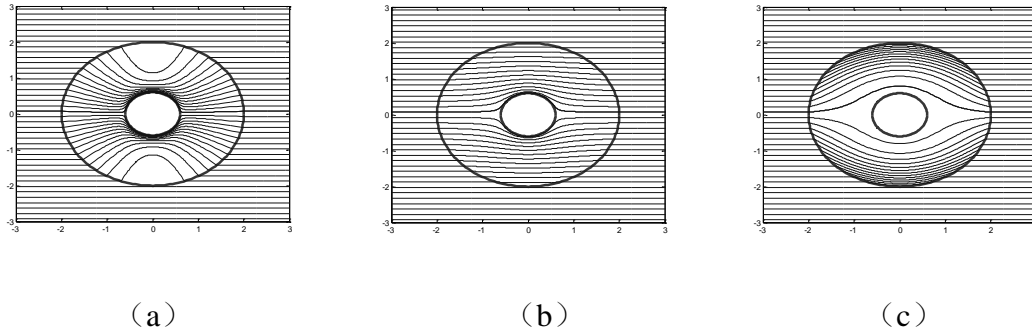
Figure 6: The propagation tracing of acoustic wave for different α in power-law density pentamode cloak
（a）α= -5；　（b）α=0；　（c）α=5

Figure 7: The acoustic field of the different α in power-law density pentamode cloak(ka=13, f=1600Hz);
（a）α= -5；　（b）α=0；　（c）α=5

4. conclusion

To summarize, we have studied the scattering characteristics of cylindrical pentamode cloaks. The TSCS of cloaks designing by different transformation function that Norris puts forward are compared. The result shows that, the propagation tracing of acoustic wave, or the distribution of wave field in the cloak shell affects the scattering property of the cloak. Decrease the wave near the inner surface of the cloak shell can improve the stealth performance of the pentamode cloak. Although the reason need to be studied further, the analysis may be helpful in optimizing the design of multilayered pentamode cloak.

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