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DISPERSION CURVES OF I-PROFILE BEAMS BY A FINITE ELEMENT METHOD

Leon GAVRIĆ ; *CETIM - Département Acoustique, BP 67 - 60304 Senlis, France*

1. INTRODUCTION

Propagation phenomena play an important role in the dynamic behaviour of large complex structures. In many physical situations, vibrations transmitted along a structural framework from a remote part of a structure or machine, can cause undesired vibration and sound radiation far away from the excitation area. Very often, the mechanical energy is transferred by straight, weakly damped structural elements of constant cross-section, which act as one-dimensional waveguides. Their dynamic properties, described by dispersion curves, are most often evaluated using one of the beam theories (Euler-Bernoulli, Timoshenko, etc.). The most important hypothesis of all the beam theories is that the cross-section stays undeformed, while undergoing vibrational movement. This hypothesis is valid at low frequencies and for compact cross-sections (large ratio of area and moment of inertia). However, where thin-walled beams are concerned, even a relatively low frequency excitation can produce transfer of mechanical energy by propagating waves associated with deformed cross-section modes. The analytical methods applicable to such deformed cross-section modes are limited to simple cross-section geometry (thin-walled circular cylindrical shell, plate strip, etc.). The paper deals with a finite element method for the computation of the propagational wavenumbers and modes of a thin-walled beam (waveguide). The method is well suited to the analysis of both undeformed and deformed cross-section modal shapes. The cross-section of the beam is modelled by using flat, thin-shell finite elements with four degrees of freedom per node. Elements have both the flexural and the membrane stiffness and inertial properties.

2. VIRTUAL WORK FORMULATION

The formulation of equations of motion is based on the virtual work principle. In wiew of the problem considered, a specific displacement field $\tilde{u}_i(x,y,z,t)$ of the following type will be investigated:

$$\tilde{u}_i(x,y,z,t) = \tilde{u}_i(y,z)e^{-j\tilde{\kappa}x}e^{j\omega t} \tag{1}$$

A point of an elastic body defined with coordinates x, y and z undergoes the steady-state harmonic motion with frequency ω, *Eq.(1)*. The elastic waves travel in the x direction with the wavenumber $\tilde{\kappa}$, while the spatial function $\tilde{u}_i(y,z)$ describes the motion of the $z-y$ plane (i.e. the plane perpendicular to the direction in wich the wave propagates), ***Kolsky*** ***[1]***. The corresponding stress $\tilde{\sigma}_{ij}(x,y,z,t)$ and strain $\tilde{\varepsilon}_{ij}(x,y,z,t)$ fields read:

$$\tilde{\varepsilon}_{ij}(x,y,z,t) = \tilde{\varepsilon}_{ij}(y,z,\tilde{\kappa})e^{-j\tilde{\kappa}x}e^{j\omega t} \tag{2.a}$$

$$\tilde{\sigma}_{ij}(x,y,z,t) = \tilde{\sigma}_{ij}(y,z,\tilde{\kappa})e^{-j\tilde{\kappa}x}e^{j\omega t} \tag{2.b}$$

These are related through the constitutive relationship $\tilde{\sigma}_{ij} = E_{ijkl}\tilde{\varepsilon}_{kl}$, where E_{ijkl} is the elasticity tensor ***Dieulesaint*** and ***Royer*** ***[2]***.

The imaginary unit is denoted by $j = \sqrt{-1}$, while complex quantities have a "tilde" sign ($\tilde{\ }$). Time is denoted by t, while x, y and z are the spatial coordinates.

The virtual work equation for such a waveguide reads:

$$\int_{\Omega}\left[\tilde{\sigma}_{ij}(y,z,\tilde{\kappa})\delta\tilde{\epsilon}^{*}_{ij}(y,z,\tilde{\kappa}) - \omega^2\rho(x,y)\tilde{u}_i(y,z)\delta\tilde{u}^{*}_i(y,z)\right]dydz = 0 \tag{3}$$

where $\rho(x,y)$ is the mass density. Virtual quantities (displacement and deformations) are denoted by δ while * denotes a complex conjugate. It should be noted that the integration is carried out over the two-dimensional domain only, while the spatial coordinate x, wich coincides with the direction of propagation, is cancelled out of the integral, in the same way as the time dependency. Similary, the mass density is also a function of only two spatial coordinates, because the cross-section is assumed to be constant along the waveguide. The first term in the equation, corresponding to the potential (elastic deformation) energy, depends on the wavenumber $\tilde{\kappa}$ while the second term, which corresponds to the kinetic energy, has no such dependence.

The general procedure for formulating finite element equations from the virtual work principle is well known, ***Zienkiewicz [3]***, and is only briefly described here. The finite element formulation used here seeks to find the displacement field $\tilde{u}_i(y,z)$ and the scalar $\tilde{\kappa}$ which satisfy the above virtual work equation. The finite element equations are obtained by discretizing the domain Ω into elements and approximating the displacement distribution within each element. For an arbitrary set of values of virtual displacements, this procedure then leads to a complex set of linear algebraic equations of the form.

$$\left([\tilde{K}(\tilde{\kappa})] - \omega^2[M]\right)\{\tilde{U}\} = 0 \tag{4}$$

where $[\tilde{K}(\tilde{\kappa})]$ is an analogon of a stiffnes matrix, which corresponds to the elastic energy in the system, $[M]$ is an analogon of a mass matrix, which accounts for the influence of the inertial forces, and $\{U\}$ is a displacement vector. The stiffness matrix $[\tilde{K}(\tilde{\kappa})]$ depends on the wavenumber $\tilde{\kappa}$. The numerical problem here consists in finding the set of generally complex-valued scalars $\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots\tilde{\kappa}_i, \ldots$ and the set of corresponding complex vectors $\{\tilde{U}_1\}, \{\tilde{U}_2\}, \ldots\{\tilde{U}_i\}, \ldots$ for a given excitation frequency ω. If the scalar parameters obtained are purely real, the corresponding vectors are real, too. In such a case, the real scalars are the wavenumbers of propagating wave fields κ_i, while the corresponding real vectors $\{U_i\}$ describe the modes of the cross section of the waveguide. The complex solutions for the wavenumber correspond to the exponentially decaying near fields, which generally do not transport any appreciable mechanical energy, unless the length of the waveguide is small or the frequency low.

3. THIN-SHELL FINITE ELEMENTS FOR ONE-DIMENSIONAL PROPAGATION

The geometry of the thin-shell propagative element is fully defined by two nodes, denoted by 1 and 2, and its thickness h, ***Fig.1***. Within the $y-z$ plane the motion of the cross-section can be described by real interpolation functions. Thin-shell flat elements have four degrees of freedom for a node n: the three displacements $\tilde{u}_n$, $\tilde{v}_n$ and $\tilde{w}_n$ in the three direction of the local coordinate axes and the rotation $\tilde{\theta}_n$ about the direction of propagation (the x-axis). The global coordinate system is chosen so that the global X axis is parallel with the local x axes of the elements. Consequently the local $y-z$ plane is parallel with the global $Y-Z$ plane of an element assembly, ***Fig.1***.

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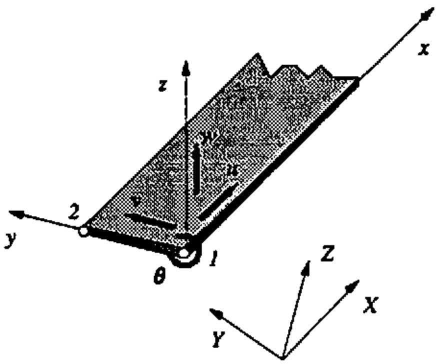

Fig. 1. *Thin-shell finite element of cross-section for analysis of propagation*

The displacement field for the element should now be specified. Strains in the direction normal to the mid-surface will be assumed to be negligible according to Kirchhoff's hypothesis, and the displacement throughout the element, $\tilde{u}(x,y)$, $\tilde{v}(x,y)$ and $\tilde{w}(x,y)$, will then be taken to be uniquely defined by three Cartesian components of the displacement, $\tilde{u}_n$, $\tilde{v}_n$ and $\tilde{w}_n$ and of the rotation about the local x-axis $\tilde{\theta}_n$ of the midsurface nodes n, $(n = 1,2)$. In order to be consistent with the shell assumptions, rotation about the local z axis is not taken as a degree of freedom, while rotation about the local y axis is an unknown quantity defined by the wavenumber $\tilde{k}$ for a given wave amplitude. As has already been mentioned, the membrane displacements, $\tilde{u}(x,y)$, $\tilde{v}(x,y)$ and the transversal deflection, $\tilde{w}(x,y)$ are treated separately. Each of these depends only on corresponding nodal degrees of freedom: the in-plane or membrane displacement fields, $\tilde{u}(x,y)$, $\tilde{v}(x,y)$, involve nodal displacements $\tilde{u}_n$ and $\tilde{v}_n$, while the lateral deflection, $\tilde{w}(x,y)$, involves the nodal degrees of freedom corresponding to the flexion, $\tilde{w}_n$ and $\tilde{\theta}_n$. In-plane displacements vary linearly within the local y axis, which assures continuity of displacements in the nodes or only C^0 continuity. To be consistent with Kirchhoff's thin shell hypothesis, the lateral deflections must be interpolated with the C^1 continuity, or the continuity of the displacement and its first derivative, within the local y-axis. The Hermite's polynomes, or so called "static beam functions" can accomplish the latter requirements, *Batoz* and *G. Dhatt [4]*. The propagative nature of the displacement field in the x direction is generated by multiplying the interpolation functions in the y direction by a complex exponential $e^{-j\tilde{k}x}$. To assure the quadrature between two in-plane displacements, the displacement component $\tilde{u}(x,y)$ is multiplied by the imaginary unit j,

$$\tilde{u}(x,y) = je^{-j\tilde{k}x}\sum_{n=1}^{2} N_n(y)\tilde{u}_n \tag{5.a}$$

$$\tilde{v}(x,y) = e^{-j\tilde{k}x}\sum_{n=1}^{2} N_n(y)\tilde{v}_n \tag{5.b}$$

$$\tilde{w}(x,y) = e^{-j\tilde{k}x}\sum_{n=1}^{2} H_n^0(y)\tilde{w}_n + e^{-j\tilde{k}x}\sum_{n=1}^{2} H_n^1(y)\tilde{\theta}_n \tag{5.c}$$

For thin shells, the transverse shear strains, and therefore the transverse shear strain energy, are negligible compared with the bending and membrane energy. The stiffness and mass matrices involve integrals over the length l of the element, which are generally of the form:

$$[\tilde{k}(\tilde{\kappa})] = \int_l [\tilde{B}^*(\kappa)]^T [D][\tilde{B}(\kappa)]dx \quad ; \qquad [m] = \int_l [\tilde{H}^*]^T h\rho[\tilde{H}]dx \qquad (6.a, 6.b)$$

where $[\tilde{B}(\tilde{\kappa})]$ and $[\tilde{H}]$ relate the deformations and the displacements within the element to the nodal displacements $\{\tilde{u}\}$, ρ is the mass density, h is the thickness of the element and $[D]$ is the elasticity matrix. The elementary matrices $[\tilde{B}(\tilde{\kappa})]$, $[\tilde{H}]$ are complex-valued matrices while the element mass matrix $[m]$ is a real-valued matrix. The matirial losses are taken to be negligible, which results in real-valued elasticity matrix $[D]$. It comes out that the complex stiffnes matrix $[\tilde{k}(\tilde{\kappa})]$ could be written in a form of a matrix polynomial, as follows.

$$[\tilde{k}(\tilde{\kappa})] = \tilde{\kappa}^4[k_4] + \tilde{\kappa}^2[k_2] + \tilde{\kappa}[k_1] + [k_0] \qquad (7)$$

where $[k_0]$, $[k_1]$, $[k_2]$ and $[k_4]$ are real-valued submatrices. Thus, the complex stiffness matrix becomes complex only when the wavenumber $\tilde{\kappa}$ is also complex-valued. The submatrices $[k_2]$ and $[k_0]$ involve the potential energies resulting from both the membrane and the bending of elastic deformations, while $[k_4]$ corresponds only to bending and $[k_1]$ only to membrane deformation energy. The stiffness matrices and the mass matrix of the element must be transformed to the global coordinate system, and assembled in order to obtain global stiffness matrices $[K_4]$, $[K_2]$, $[K_1]$ and $[K_0]$ and a global mass matrix $[M]$. It should be noted that the transformation of the elementary matrices involves only a two-by-two matrix of direction cosines between the y, z and Y, Z axes, since the x and X axes are parallel.

4. EQUATIONS OF MOTION OF ASSEMBLED FEM MODEL

The equations of motion of an assembly of finite elements described previously takes the following form.

$$\left(\tilde{\kappa}^4[K_4] + \tilde{\kappa}^2[K_2] + \tilde{\kappa}[K_1] + [K_0] - \omega^2[M]\right)\{\tilde{U}\} = \{0\} \qquad (8)$$

where for a given excitation frequency ω, $\tilde{\kappa}$ is the wavenumber, and $\{\tilde{U}\}$ is an unknown eigenvector describing the corresponding mode. Generally, both of these quantities are complex. The real-valued wavenumbers characterize the propagative displacement field, which transmits mechanical energy within the waveguide. The corresponding modes are also real.

To solve *Eq.(8)* we must transforme it in a more appropriate form. The procedure starts with the inversion of that part of *Eq.(8)*, which is not dependant on the wavenumber, $([K_0] - \omega^2[M])$. For some excitation frequencies $\omega^2 = \Omega_i^2$, the matrix to be inverted is singular. These frequencies, which are called cut-on or critical frequencies, can be computed by solving the simple eigenvalue problem $([K_0] - \Omega^2[M])\{U\} = \{0\}$, i.e. by letting $\tilde{\kappa} = 0$ in *Eq.(8)*. Since the part of matrix equation to be inverted is very ill-conditioned in the vicinity of critical frequencies, $det|[K_0] - \omega^2[M]| \sim 0$ when $\omega \rightarrow \Omega_i$, the most accurate matrix inversion procedures should be chosen. *Eq.(8)* is then multiplied by the inverted matrix and divided by $\tilde{\kappa}$, yielding the following relationship:

$$\left([A_1] + \tilde{\kappa}[A_2] + \tilde{\kappa}^3[A_4]\right)\{\tilde{U}\} = \frac{1}{\tilde{\kappa}}\{\tilde{U}\} \qquad (9)$$

where $[A_n] = -([K_0] - \omega^2[M])^{-1}[K_n]$; $n = 1, 2, 4$. By adding the three following identities $\tilde{\kappa}^{n-1}\{\tilde{U}\} = \frac{1}{\tilde{\kappa}}\tilde{\kappa}^n\{\tilde{U}\}$ for $n = 1, 2, 3$ to the system of equations, *Eq.(9)* can be extended to a simple eigenvalue problem.

$$\begin{pmatrix} [A_1] & [A_2] & [0] & [A_4] \\ [I] & [0] & [0] & [0] \\ [0] & [I] & [0] & [0] \\ [0] & [0] & [I] & [0] \end{pmatrix} \begin{pmatrix} \{U\} \\ \tilde{\kappa}\{\tilde{U}\} \\ \tilde{\kappa}^2\{\tilde{U}\} \\ \tilde{\kappa}^3\{\tilde{U}\} \end{pmatrix} = \frac{1}{\tilde{\kappa}} \begin{pmatrix} \{\tilde{U}\} \\ \tilde{\kappa}\{\tilde{U}\} \\ \tilde{\kappa}^2\{\tilde{U}\} \\ \tilde{\kappa}^3\{\tilde{U}\} \end{pmatrix} \qquad (10)$$

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where $[I]$ is the identity matrix. The unknown eigenvector to be computed, contains not only the displacement field $\{\tilde{U}\}$ but also its first, second and third "derivatives" with respect to the X axis, contained in $\tilde{\kappa}\{\tilde{U}\}$, $\tilde{\kappa}^2\{\tilde{U}\}$ and $\tilde{\kappa}^3\{\tilde{U}\}$. The dimension of the unknown eigenvector is four times the dimension of the finite element model, which corresponds to the number of solutions for the eigenvalue (inverse of the wavenumber).

5. COMPUTATION EXAMPLES

5.1 Circular cylindrical shell

The finite element results and analytic results using Donnell's thin shell theory, *Fuller [5]*, are compared for a 2 mm thick steel pipe of 100 [mm] diameter. Young's modulus is taken to be 210 000 [MPa], Poisson's ratio 0.31, while the mass density is 7 800 [kg/m^3]. The pipe is modelled using 48 nodes and 48 propagative thin-shell finite elements, described previously. The finite element model has 192 degrees of freedom.

Excitation Frequency	method	axial	torsional	mode 1	mode 2	mode 3
500 [Hz]	analytic	0.605	0.980	4.16	-/-	-/-
	F E M	0.604	0.979	4.25	-/-	-/-
1000 [Hz]	analytic	1.21	1.96	6.16	8.71	-/-
	F E M	1.21	1.96	6.19	9.64	-/-
1500 [Hz]	analytic	1.82	2.94	7.79	12.4	-/-
	F E M	1.82	2.94	7.80	12.8	-/-
2000 [Hz]	analytic	2.42	3.92	9.26	15.1	14.5
	F E M	2.42	3.92	9.25	15.3	16.5
2500 [Hz]	analytic	3.03	4.90	10.6	17.5	20.5
	F E M	3.03	4.90	10.6	17.6	21.3
3000 [Hz]	analytic	3.64	5.88	11.9	19.6	24.6
	F E M	3.64	5.88	11.3	19.8	25.1

Fig. 2. *Wavenumbers of circular cylindrical shell obtained analyticaly and by FEM*

The comparison between the wavenumbers is given in *Fig.2*.

Agreement is fairly good except in the viscinity of the cut-on frequencies where $([K_0] - \omega^2[M])$ becomes very ill-conditioned. The modes of the cross-section computed using the finite element model are shown in *Fig.3*.

5.2 I - profile beam

The thin-walled beam profile, often used as a structural framework element, and called I-profile, is analysed using a mesh of 36 thin-shell finite elements and 37 nodes. The finite element model has 148 degrees of freedom. The I-profile is 180 [mm] × 180 [mm], while the thickness of the wall is 4 [mm]. The Young's modulus is 210 000 [MPa], Poisson's ratio is 0.3, and the mass density is 7 800 [kg/m^3]. The complex wavenumbers are computed for the excitation frequencies from 0 to 1000 [Hz]. Then only those propagative (real) are then given in a form of dispersion curves in *Fig.4*.

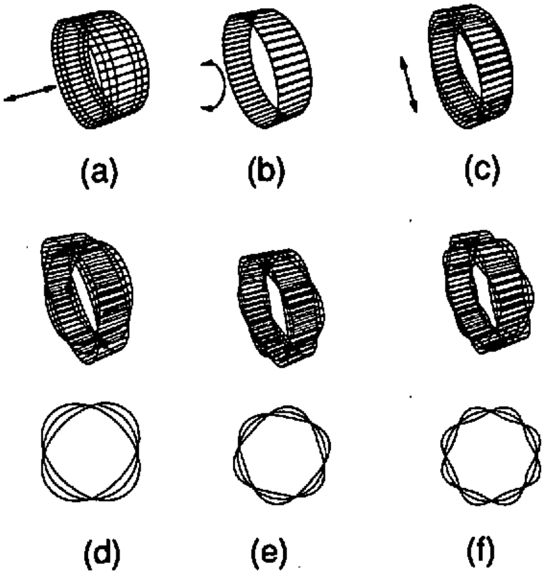

Fig. 3. *Modes of cross-section of circular cylindrical shell (modes 0-4)*

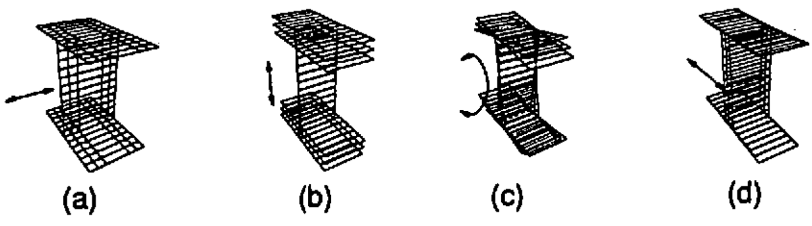

Fig. 4. *Undeformed propagatif modes of I-profile beam ; frequency of excitation 50 [Hz]*

The low frequency results of the finite element analysis, described previously, match the simple Euler-Bernoulli beam theory well. For the excitation frequency of 50 [Hz] four "beam" modes of the beam cross-section are extracted: the axial mode, two bending modes and the "torsional" mode. The wavenumbers computed using the beam theory, which correspond to the axial mode, *Fig.4.- (a)*, and two bending

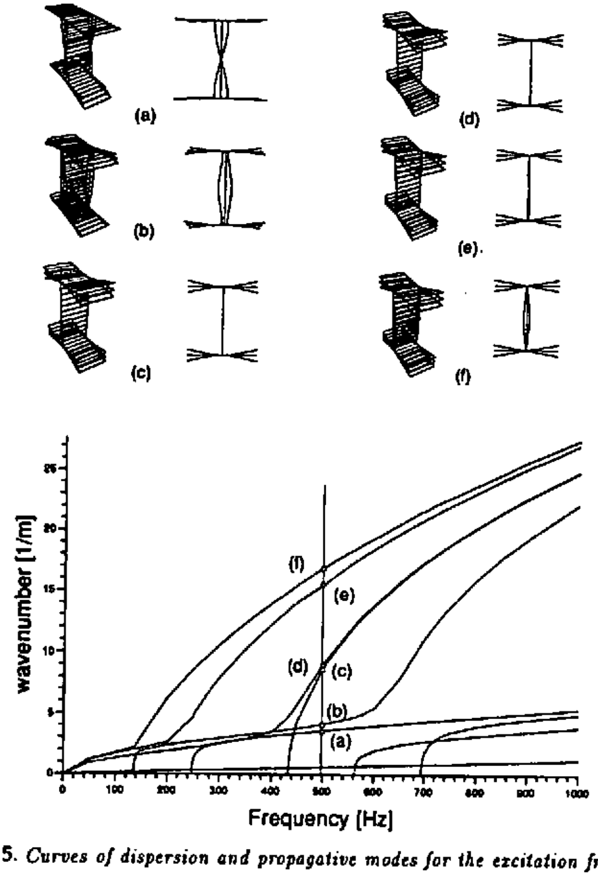

Fig. 5. *Curves of dispersion and propagative modes for the excitation frequency of 500 [Hz]*

modes, *Fig.4.- (b)*, and *Fig.4.- (d)*, are 0.061 [1/m], 0.874 [1/m] and 1.19 [1/m] respectively, while the finite element computation gives 0.062 [1/m], 0.873 [1/m] and 1.13 [1/m].

At higher frequencies the propagative modes become "deformed". The only propagative "undeformed" mode is the axial mode. With the increasing of the excitation frequency the other deformed modes started to propagate energy (cut-on frequency phenomenon). For the excitation frequency of 500 [Hz] except the axial mode there are another 6 propagative modes, *Fig.5*. The associated wavenumbers are 3.60 [1/m], 4.13 [1/m], 8.79 [1/m], 9.08 [1/m], 15.7 [1/m] and 16.9 [1/m], which do not match any of the bending wavenumbers computed by the beam theory: 2.76 [1/m] and 3.77 [1/m].

The modes in *Fig.5.- (c)* and in *Fig.6.- (d)* have nearly the same wavenumbers (8.79 [1/m] and 9.08 [1/m]) but different mode shapes. The modal motions of the two horizontal plate strips are out of phase (180°) in *Fig.5.- (c)*, while in the mode in *Fig.6.- (d)* they are in phase (0°). The existence "deformed modes of the cross-section", involved in the transfer of energy along the thin-walled waveguides, makes the simple beam approach completely unsatisfactory for the analysis of energy transfer (and the analyse of dynamic response) of thin-walled waveguides at higher frequencies.

6. CONCLUDING REMARKS

The finite element method presented in the paper is used to compute the dispersion properties of thin-walled waveguides, i.e. the wavenumbers and associated cross-section modes. The finite element results coincide with the results obtained by the Euler-Bernoulli beam theory at low frequencies. The finite element analysis of the thin-walled I-profile beam shows that the beam theories (Euler-Bernoulli, Timoshenko ...) become inaccurate at higher frequencies due to deformation of the cross-section during the vibrational movement.

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