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INTERACTION OF NONLINEAR NORMAL MODES OF A CANTILEVER BEAM

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In the current study, real normal mode manifold approach is used to determine the nonlinear normal modes of the cantilever beam governed by a cubic order geometric and inertial nonlinearity. The system of governing partial differential equations are discretized by expanding the beam displacement using a series of modes which are the Eigenfunctions satisfying the linear beam equation along with the boundary conditions for a cantilever beam. The nonlinear normal modes are then determined by rewriting the discretized system of equation as two first order system of equations in terms of displacement and velocity. The method of multiple scales is adapted as solution procedure to derive the two first-order nonlinear ordinary-differential equations governing the modulation of the amplitudes and phases of the interacting modes of a cantilevered beam. The numerical results show how energy imparted to the first mode get exchanged with other modes through the nonlinear interactions. The bar chart of the logarithmic of generalized coordinate is analyzed for the first ten modes. The nonlinear frequency response diagrams of the cantilever beam for the primary resonance are obtained. The stable and unstable portions of each frequency-response curves are then determined. The influences of excitation level and nonlinearity on the nonlinear frequency response curves have been inspected.

Keywords: nonlinear normal mode, modal interaction, stability, cantilever beam

1. Introduction

The physical phenomenon like chaotic behavior, jump phenomenon, combination of resonances and modal interactions due to nonlinearity in structural dynamics are always challenging to researcher investigating nonlinear structural dynamics. The cantilever beam is the fundamental element of the engineering structure which could be simplified model of many complex structure like helicopter rotor blades, compressor or turbine blade, antennas used on satellite or spacecraft, airplane wings, gun barrels, robot arms, high-rise buildings and many more. Hence, studying the dynamic response of such simple configuration will help us to understand the behavior of the complex structure.

Various researcher have investigated nonlinear dynamic behavior of the cantilever beam using different techniques for many years. The influence of modal interaction on the nonlinear response of structure with quadratic nonlinearities is reviewed theoretically and experimentally by Nayfeh and Balachandran [1]. In reference [2], Arafat et al. studied bifurcation and chaos of the non-planar responses of metal inextensional cantilever beams with parametric excitation. In ref. [3], Malatkar studied theoretically and experimentally the nonlinear response of a flexible cantilever beam to a harmonic excitation, and demonstrated that the energy can transfer from the third mode to the first mode.

Lee et al. carried out the stability analysis of non-planar free vibrations of a cantilever beam using the nonlinear normal mode concept [4]. Even though there is great progress in research on nonlinear dynamics of cantilever beam, the study of nonlinear modal interactions and the stability of nonlinear normal mode is still limited.Generally, the nonlinear vibration of a cantilever beam is theoretically analyzed using two approaches. One approach is to solve the governing partial differential equations using perturbation methods. Another way is to use Galerkin method to spatially discretize the partial differential equation and then use perturbation methods. In the present study, the system of governing partial differential equations are discretized by expanding the beam displacement using a series of modes which are the Eigenfunctions satisfying the linear beam equation along with the boundary conditions for a cantilever beam. The method of multiscale is used to derive the first order nonlinear ordinary-differential equations governing the modulation of the amplitudes and phases of the interacting modes of a cantilevered beam. Then the modulation equations are solved numerically to study the stability of the nonlinear normal modes.

2. Governing Equation

The equation of motion governing an Euler-Bernoulli clamped-free uniform beam under large deflection using inextensionality constraint is derived in [5]. The non dimensional form of equations governing the motion of a cantilever beam is given by

$$\ddot{w} + w'''' + \Big(w'''w'^2 + w'w''^2\Big)' + \Big[w'\Big(\int_1^x \int_0^x \ddot{w'^2} dx dx\Big)\Big]' = 0. \tag{1}$$

The associated boundary conditions are

$$w = w' = 0 \ \text{ at } \ x = 0 \qquad \text{and} \qquad w'' = w''' = 0 \ \text{ at } \ x = 1. \tag{2}$$

The transverse displacement of beam w is nondimensionalized by the beam length L and time is nondimensionalized by $L^2\sqrt{\frac{\rho A}{EI}}$ where EI is the rigidity of the beam, ρ is the beam density per unit length, and A is the beam cross-section area. The over dot and prime indicate differentiation with respect to time t and distance along the beam x respectively. The third term in the equation (1) is cubic order geometrical nonlinearity that is consequences of non-linear dependency of curvature on the w. The fourth term in the equation is non-linear inertial term.

The normal modes of linear system can have two distinct properties: invariance property and modal superposition. The concept of modal superposition cannot be apply for nonlinear system however the concept of invariance property can be extended to nonlinear system. In terms of the system's phase space, normal mode motion occurs on a two-dimensional subspace which is invariant. In ref. [5], Hsieh et al. extended the concept of a normal mode motion occurring on an invariant two-dimensional subspace to nonlinear system defined in equation (1) - (2). These subspaces are two-dimensional planes for linear systems where as for nonlinear system, these are curved and are referred to as the nonlinear normal mode manifolds. This manifold passes through stable equilibrium point $(w, \dot{w})$ and is tangential to the Eigen spaces of the associated linearized system at that point. The response of entire system is described in terms of response of some single point $x = x_0$ through amplitude dependent mode shape as

$$w(x_0, t) = W(w_0(t), \dot{w}_0(t), x, x_0), \qquad \dot{w}(x_0, t) = \dot{W}(w_0(t), \dot{w}_0(t), x, x_0), \tag{3}$$

where $w_0(t), \dot{w}_0(t)$ are displacement and velocity of point x_0.

The authors used equation (3) as geometric constraints to eliminate time derivatives form the equation of motion and then obtained nonlinear normal modes through asymptotic expansion of resultant spatial equations. However functions W and $\dot{W}$ valid at a point x_0 and may not be valid globally. The selection of the point x_0 makes this method cumbersome. To overcome the shortcoming of the above method, Nayfeh et al. [6], and Nicolas et al. [7] proposed to first discretize the system using Galerkin approach. Nayfeh et al. in [6] demonstrated equivalence of several methods like invariant manifold method, perturbation methods applied to discretize system and multiple scale method applied directly to equation of motion to study non-linear modes of continuous system with inertia and geometric nonlinearity. In ref. [8], Nayfeh et al. showed that the method of multiple scales applied directly to the governing partial differential equation and boundary conditions is equivalent to discretization of the original problem to obtain an approximate non-linear modes and natural frequencies of cantilever beam with cubic nonlinearity.

The continuous system given in (1) is discretized by expanding beam displacement using a series of linear modes. Each of these modes has a modal amplitude called generalized coordinate which is function of time [9]. In modal analysis, the total displacement of the beam is expressed as

$$w(x,t) = \sum_{i=1}^{n} q_i(t)\phi_i(x), \tag{4}$$

where, n is the number of modes and $q_i(t)$ are the unknown generalized coordinates. $\phi_i(x)$ are the beam Eigenfunctions obtained by solving the equation (1) neglecting the nonlinear term and satisfying the boundary conditions (2). The Eigenfunctions are (5)

$$\phi_i(x) = \cosh(\alpha_i x) - \cos(\alpha_i x) - \beta_i[\sinh(\alpha_i x) - \sin(\alpha_i x)], \tag{5}$$

where α_i are the roots of the equation

$$\cos\alpha_i \cosh\alpha_i + 1 = 0 \qquad \text{and} \qquad \beta_i = \frac{\cosh(\alpha_i) + \cos(\alpha_i)}{\sinh(\alpha_i) + \sin(\alpha_i)}. \tag{6}$$

The solution of equation (6) is available in literature [9]. $\omega_i = \alpha_i^2$ are the corresponding non dimensional natural frequencies of the beam in bending. Useful mathematical property of the Eigenfunctions is that they form a set of orthonormal mathematical functions, which means

$$\int_0^1 \phi_i(x)\phi_j(x)dx = \delta_{ij}, \qquad \int_0^1 \phi_i^{iv}(x)\phi_j(x)dx = \omega_i^2\delta_{ij}, \tag{7}$$

where δ_{ij} is the Kronecker delta function. Substituting the general solution (4) into (1) and multiplying byϕ_i then integrating over the length of the beam.

$$\ddot{q}_i + \omega_i^2 q_i + N_i(q,\dot{q},\ddot{q}) = 0 \tag{8}$$

For the case of cubic geometric and inertia nonlinearity, the nonlinearity can be expressed as [6]

$$N_i(q,\dot{q},\ddot{q}) = N1_{ij}q_j^3 + N2_{ij}q_j\dot{q}_j^2 + N2q_j^2\ddot{q}_j \tag{9}$$

where, $$N1_{ij} = \int_0^1 [\phi_i'(\phi_i''\phi_i')']'\phi_j dx \;\text{ and }\; N2_{ij} = \int_0^1 [\phi_i' \int_1^x \int_0^x \phi_i'^2 dx dx]'\phi_j dx. \tag{10}$$

Using (8) $$N_i(q,\dot{q},\ddot{q}) = (N1_{ij} - \omega_j^2 N2_{ij})q_j^3 + N2_{ij}q_j\dot{q}_j^2 \tag{11}$$

The nonlinear normal modes of the cantilever beam are determined using the real normal mode manifold approach. The procedural details of the method is elaborately given in [6]. The k^{th} nonlinear mode obtained by treatment of the discretized system can be written as

$$w_k(x,t) = \phi_k(x)q_k(t) + \sum_{j\neq k}^{N} \phi_k[H1_{jk}q_k^3 + H2_{jk}q_k\dot{q}_k^2] + \dots \tag{12}$$

where,

$$H1_{ij} = \frac{(7\omega_k^2 - \omega_j^2)N1_{jk} - (5\omega_k^2 - \omega_j^2)\omega_k^2 N2_{jk}}{(\omega_k^2 - \omega_j^2)(9\omega_k^2 - \omega_j^2)} \text{ and } H2_{ij} = \frac{(6N1_{jk} - (3\omega_k^2 + \omega_j^2)N2_{jk}}{(\omega_k^2 - \omega_j^2)(9\omega_k^2 - \omega_j^2)} \quad (13)$$

Every point of the system reaches its maximum at the same instant. Thus, the mode shape $W_k(x)$ is defined by

$$W_k(x) = \phi_k(x)q^* + \sum_{j\neq k}^{N} \phi_k[H1_{jk}q_k^{*3} + \dots \quad (14)$$

where q^* is the maximum value. The first four nonlinear normal modes are given in figure 1 along with linear normal modes in section 3.1.

The modal interaction nonlinear vibration when system is excited by external harmonic excitation of farm $P = P_0cos(\Omega t)$ is studied. The discretized equation can be expressed as

$$\ddot{q}_i + \alpha_i^4 q_i = -\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{n}\Big[N1_{jkl}^i q_k q_l + N2_{jkl}^i(\ddot{q}_k q_l + \dot{q}_k \dot{q}_l)\Big]q_j + P_i \quad (15)$$

for $i = 1, 2, 3, \dots n$, where n is the number of modes and sufficient number of modes are considered so that the series is converged. In the current calculation, summation over 10 modes are considered. And $N1_{ij}$, $N1_{ij}$ and P_i are defined as

$$N1_{jkl}^i = \int_0^1 [\phi_j'(\phi_k''\phi_l')']'\phi_i dxdx \quad N2_{jkl}^i = \int_0^1 \left[\phi_j' \int_1^x \int_0^x \phi_k'\phi_l' dxdx\right]' \phi_i dx \quad P_i = \int_0^1 P\phi_i dx. \quad (16)$$

Numerical integration is carried out using Simpson's $1/3^{rd}$ rule to calculate M_{jkl}^i and N_{jkl}^i matrices. The nonlinear modal interactions is studied using prescribed initial conditions $q_i(t = 0) = \delta_{i1}$ and $\dot{q}_i(t = 0) = 0$. The results are presented in the section 3.2.

Now, The perturbation analysis is carried out on the nonlinear equation to understand the local stability of solution. The multiple scale method is adapted as solution procedure to investigate the case of primary excitation $(\Omega \approx \omega)$ of the system. At primary excitation, small magnitude of external excitation can cause comparatively large response of system due to primary resonance. Therefore, in order to study primary resonance(weak excitation) of the system, equation (8) using (11) can be written in the form (17) assuming non-linear terms to be at the same level of approximation as the external forcing. Therefore, a small dimensionless parameter ϵ as a book keeping device is introduced. It is assumed that the beam vibrates predominantly in in the first mode in the range of frequencies considered in this study. Hence, only the first mode is considered.

$$\ddot{q} + \omega^2 q = \epsilon(P_0 cos\Omega t - \gamma q^3 - cq\dot{q}^2) \quad (17)$$

where $\gamma = (N1_1 - \omega_1^2 N2_1), c = N2_1$. In the method of multiple scale, solution is expanded as a function of multiple independent time variable, in power series of small parameter (ϵ). In this method solution is assumed to be uniform expansion of the form

$$q(t, \epsilon) = q_0(T_0, T_1) + \epsilon q_1(T_0, T_1) + \mathcal{O}(\epsilon^2) \quad (18)$$

The expansion is carried out to the second order of ϵ. In the equation q_0 and q_1 are functions to be determined and $T_1 = t$ is a fast scale characterizing motions occurring at one of the natural frequencies and $T_1 = \epsilon t$ is the slow scale characterizing the shift in the natural frequencies due to non-linearity. The time derivative in terms of T_n are given $\frac{d}{dt} = D_0 + \epsilon D_1 + ...$

Substituting (18) into (17) and equating to zero the coefficients of like powers of ϵ yields

$$\epsilon^0 : (D_0^2 + \omega^2)q_0 = 0, \text{ and } \epsilon^1 : (D_0^2 + \omega^2)q_1 = P_0 cos(\Omega T_0) - 2D_0D_1q_0 - \gamma q_0^3 - cq_0(D_0q_0)^2 \quad (19)$$

Equation to the order ϵ^0 is the linear decoupled free vibration equations of the beam. Considering boundary conditions of the cantilever beam, the general solution to the such equation is of form

$$q_0 = A(T_1)e^{i\omega T_0} + \bar{A}(T_1)e^{-i\omega T_0}. \quad (20)$$

where A is an unknown complex-valued function of the slow time scale and $\bar{A}$ is the complex conjugate of A_i, which are to be determined. The detuning parameter, σ, is introduced to convert the near resonant term to resonant one. Letting $\Omega = \omega + \epsilon\sigma$, using $T_1 = \epsilon t$, we can write $\Omega T_0 = \omega T_0 + \sigma T_1$. Substituting (20) into equation to the order ϵ^1 and equating the coefficient of e^{iwT_0} in the resulting equation to zero to meet solvability condition [10]

$$\frac{P_0}{2} e^{i\sigma T_1} - 2i\omega A' - (3\gamma + c\omega^2)A^2\bar{A} = 0. \tag{21}$$

To solve equation (21), let $A = \frac{1}{2}a(T_1)e^{i\theta(T_1)}$. Substituting into equation (21) and separating real and imaginary parts, the following pair of modulation equations are obtained.

$$a' = \frac{P_0}{2\omega}\sin\psi \qquad \text{and} \qquad \omega a\psi' = \sigma\omega a - \frac{a^3}{8}(3\gamma + c\omega^2) + \frac{P_0}{2}\cos(\psi). \tag{22}$$

where ψ = σ T_1 - θ and ψ' = σ - θ'. Squaring and adding the two equations (with $a' = 0$; $\psi' = 0$), a is given by the implicit solution to the following frequency response equation.

$$a\Big[2\sigma\omega - \frac{a^2}{4}(3\gamma + c\omega^2)\Big] = \pm P_0 \tag{23}$$

inserting $\sigma = \Omega - \omega$ and ignoring ϵ which merely serves the purpose of indicating the level of approximation to study variation of amplitude with the excitation frequency.

$$\frac{\Omega}{\omega} = 1 + \frac{a^2}{8\omega^2}(3\gamma + c\omega^2) \pm \frac{P_0}{2a\omega^2} \tag{24}$$

The stability of solution is determined by evaluating the Eigenvalues of the Jacobian of the equations (22) at the singular point($\bar{a}$, $\bar{\psi}$). The Eigenvalues (λ_i) of the Jacobian matrix is found that

$$\lambda_{1,2} = \pm\sqrt{-\Big[\sigma - \frac{a^2}{8\omega}(3\gamma + c\omega^2)\Big]\Big[\sigma - \frac{3a^2}{8\omega}(3\gamma + c\omega^2)\Big]}. \tag{25}$$

A solution $\bar{a}$ is unstable if at least one eigenvalue has a positive real part, otherwise it is stable. The stability plots are explained in the section 3.3

3. Results and Discussion

The results for the nonlinear normal mode and the stability analysis of a cantilever beam studied using the techniques described in the previous section is presented here. Initially, for the case of nonlinear free vibrations, the nonlinear normal modes constructed using real normal mode manifold approach are presented. Then, the nonlinear interaction of normal modes for the case of forced vibration is addressed where the plate motion is excited by the uniform external pressure having sinusoidal time dependence. Finally, the detailed study of linear stability analysis of nonlinear vibration of a cantilever beam is discussed.

3.1 Nonlinear Normal Modes

The nonlinear normal mode is a motion where every points in the system execute periodic (not necessarily harmonic) motion with the same period. Every points in the system pass through their static-equilibrium positions at the same time, and achieve their maximum displacements at the same time. It can be shown form equation (12) that at any instant of time t_0 at which $q(t_0) = 0$ then $w(x, t_0) = 0$ for all x and at any instant of time t^* at which $\dot{q}(t^*) = 0$, then $\dot{w}(x, t^*)$ is zero for all x. Moreover the corresponding value of q at t^* i.e. $q^* = q(t^*)$ and $w(x, t^*)$ are maximum. For a given q^*, which depends on the initial conditions, the nonlinear mode shapes is compared with the linear ones in figure 1 for the lowest four modes. For very low values of q^*, the nonlinear modes become close to linear mode. Therefore there is critical value of the q^* below which the construction of the nonlinear normal mode breaks down.

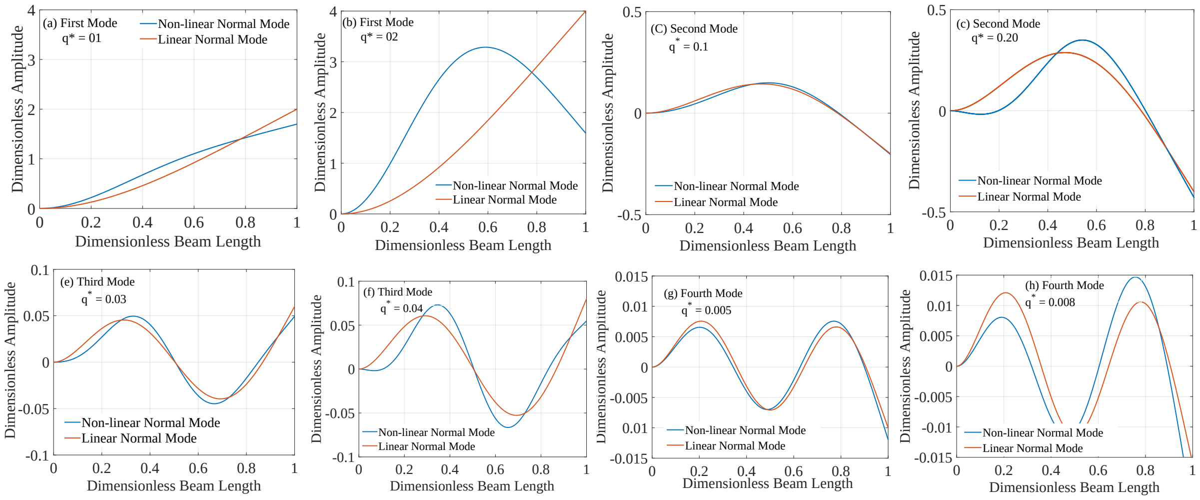
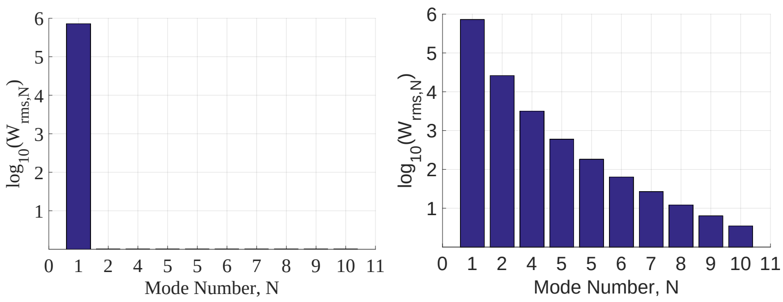
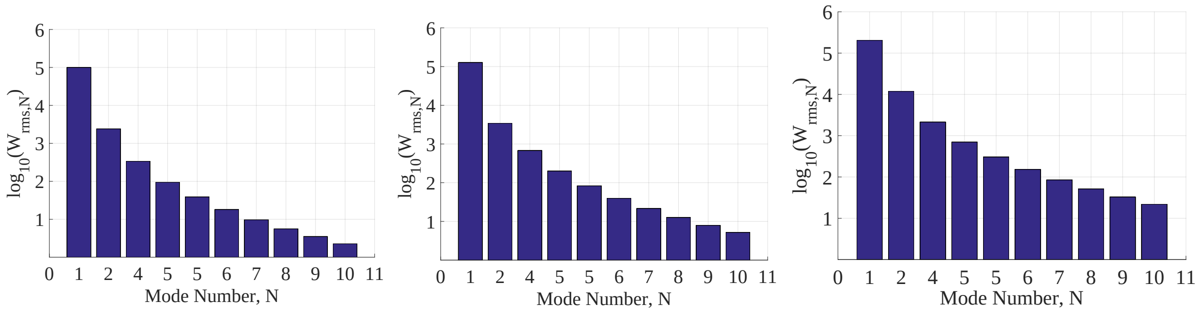


Figure 1: The comparison of linear and nonlinear modes for the lowest four modes.



(a) Linear system (b) Nonlinear System

Figure 2: Modal interaction of linear and nonlinear Vibration



(a) $P_0 = 1$ (b) $P_0 = 2$ (c) $P_0 = 5$

Figure 3: Modal Interaction of nonlinear system for different level of excitation

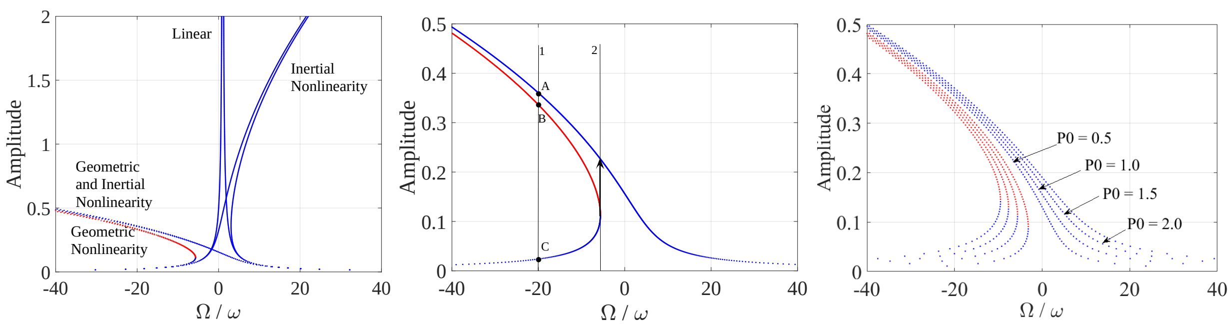
3.2 Modal interaction

The linear normal modes of linear system are uncoupled that is the energy imparted to any one mode remains in that mode. But in presence of non-linearity, normal modes are coupled. In nonlinear system, nonlinearities provide a mechanism for exchanging energy when imparted into one mode with other modes under certain conditions [10]. This situation in which energy is exchanged between modes in a system is studied here.

The bar chart of the logarithmic of generalized coordinate is plotted for the first ten modes. The figure 2(a) is the bar graph for the free linear system when the first mode is excited with the initial displacement of magnitude 1. In the response also the first mode gets excited but all other modes which indicate that the modes are decoupled. For the same initial condition, the bar chart is obtained for the free nonlinear system. From figure 2(b) it is evident that modes do interact with each other for the nonlinear system. The similar study is carried out for the nonlinear forced vibration to study the influence of P_0 on the nonlinear modal interaction. Figure 3(a) to 3(c) shows the nonlinear interaction of modes for different values of P_0. The coupling of modes becomes stronger for higher value of P_0.

3.3 The Perturbation Analysis

The behavior of cantilever beam in presence of cubic order non-linearity is presented in this section. The frequency response of the system is obtained by plotting the variation of amplitude with excitation frequency form equation 24. Figure 4(a) shows the effect of non-linearity. For $\gamma = 0$ and $c = 0$, the plot shows familiar linear resonance curve in which the peak is straight up unbounded. The plot also shows that the inertial nonlinearity causes frequency response curve bend towards right in absence of the geometric nonlinearity. The frequency response curve of geometric nonlinearity alone is much similar to that of combined inertial and geometric nonlinearity which indicate that the geometric nonlinearity has strong influence on the softning effect. The geometric nonlinearities are more important for the response of the first mode than the inertia nonlinearities and similar conclusion drwan by Nayfeh et al. in [11].



(a) Effect of non-linearity ($P_0 = 1$) (b) Jump phenomenon (c) Effect of excitation level

Figure 4: Frequency responses of nonlinear system. Red dotted line indicates unstable solutions and blue dotted line indicate the stable solutions

Figure 4(b)shows typical frequency response curve for nonlinear system at external excitation $P_0 = 1$. There are some interesting observation made on response curve. First, the response curve is unbounded due to absence of the damping. Second, The peak of the curve is bent towards left. This is characteristic of the systems with nonlinear restoring forces of the softening type ($\gamma < 0$). This contrasts the straight-up peak characteristic of linear systems. Third, there are discontinuous jumps in the response, also a consequence of the bent peak. Fourth, stability of solution can be observed in the frequency response plot. The solution to be stable, there should exist no eigenvalue with positive real part for the equation 25. The condition at which solution is unstable is met in the region of red line curve in the figure 4(b) and all other parts of the response are stable.

Figure 4(b) shows the existence of multiple solution. There are two vertical lines in the figure, marked 1 and 2. For the line 2, it is observed that there are three amplitude possible A, B and C corresponding to the single frequency. The point B is unstable and A, C are stable.

The figure also shows the phenomenon of jumps, which are discontinuous changes in the response of a system as a forcing frequency is slowly varied. When forcing frequency is increased slowely from some very low value, then the system response stays on the lower branch, as it reaches line 2, it jumps up to the higher branch, as shown by the arrow [10].

The response curves are plotted in figure 4(c) for the different level of magnitude of external excitation (P_0). Naturally, as (P_0) is reduced then so is nonlinear bend of the peak. Also, the width of the resonant region shrinks when (P_0) is reduced.

4. Conclusion

The nonlinear normal modes of cantilever beam with cubic order geometric and inertia non-nonlinearities are constructed using the real normal mode manifold approach. The governing equations are discretized by expanding the beam displacement using a series of complete set of basis functions that satisfy the boundary conditions, namely, the linear mode shapes. The behavior of nonlinear system such as multiple steady state solutions, jump phenomena and stability analysis are studied for the primary resonance. Two first-order nonlinear ordinary-differential equations governing the modulation of the amplitudes and phases of the interacting modes of a cantilevered beam are solved analytically. Stable and unstable portions are indicated in the nonlinear frequency response diagrams. The influences of level of external excitation on the nonlinear frequency response curves have been inspected. It is observed that the frequency response of the cantilever beam is strongly affected by the non-linearity and excitation level.

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