AXIAL VIBRATION CONTROL OF PROPELLER SHAFT SYSTEM USING DYNAMIC ANTI-RESONANCE VIBRATION ISOLATOR

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Harmonic axial force resulting from a propeller's first vibration mode is a major cause of tonal sound radiation of an underwater vehicle. To reduce the harmonic force, we employ a dynamic anti-resonant vibration isolator (DAVI) in parallel with thrust bearing of the shafting system to attenuate vibration transmitted to the hull. Method of transfer matrices and substructure synthesis are used to create a semi-analytical dynamic model of the propeller-shaft-hull system with DAVI. In this model, the elastic properties of the propeller and foundation are taken into consideration. The force transmissibility is then used to evaluate the isolation performance of DAVI. Finally, parametric studies are conducted to reveal the effects of the parameters of DAVI on the vibration control performance. It is demonstrated numerically that by using the DAVI, the vibration transmission to the hull is greatly attenuated at the designed frequency without obviously changing the axial fundamental resonance frequency of the shafting system.

Keywords: Dynamic anti-resonance vibration isolator; Vibration control; Longitudinal vibration; Propeller; Sound radiation

1. Introduction

In the low frequency range, the fluctuating forces of the propeller are the main source of underwater sound for a submarine [1]. Owing to the rotation of the propeller in a non-uniformity and nonstationary wake, the fluctuating forces are composed of narrow periodic components and broadband components [2,3]. Narrowband periodic force components typically occur at frequencies that are proportional to the shaft speed or number of blades due to the nonuniform inflow [4,5]. Broadband components are distributed over a bandwidth of several hundred hertz with some “hump” at the first and second natural frequency of the propeller [6,7] due to the unsteady inflow turbulence and propeller blades interactions [8].

The fluctuating forces of the propeller, especially the axial component, lead to significant underwater sound radiation when they are transmitted to the hull via the shafting system [9,10]. Although there are many investigations on the vibration control of the shafting system induced by axial force of the blade passing frequency, few studies concentrate on the vibration of marine propellers induced by the axial force of the propeller’s first natural frequency. In these researches [11,12], the propeller is often simplified as a rigid concentrated mass thus the elasticity of the propeller can not be taken into consideration. Some works [13-15] have indicated that the elastic blades subjecting to the broadband excitation would inevitably result in elastic vibrations in the propeller and its shafting system. For this reason, the elasticity of the propeller will be considered in this paper and the vibration problem induced by the propeller’s first natural frequency will be addressed utilizing an isolator called dynamic anti-resonant vibration isolator (DAVI).

DAVI was firstly proposed by Flannelly et al. [16] in the 1960s to accommodate the strict restriction on stiffness and mass of the isolator in the aerospace industry. When inertial force generated by the isolator mass cancels the spring force, the anti-resonance occurs. The frequency of the anti-resonance depends on its lever ratio, the mass of the isolator, and the stiffness of the spring. Due to the introduction of the lever, the effective mass of the system will be increased, therefore the isolator is capable of operating in a lower-frequency range. The references [17-21] presented the applications of DAVI in the aerospace industry. Halwes and Simmons [22] designed a fluid-type DAVI with a higher leverage ratio in which the isolator mass will be further reduced. In this paper, without considering its realization, the feasibility of DAVI in vibration control of the shaft at the first natural frequency of the propeller is investigated from the theoretical aspect. For this reason, a DAVI is to be installed in parallel with the thrust bearing. It is expected that by using the DAVI, the force transmitted to the hull can be significantly reduced with few effects on the axial fundamental resonance frequency of the shaft. Although DAVI is widely used in the aerospace engineering, to the authors' knowledge, vibration control of the propeller-shaft-hull system at the first natural frequency of the propeller using DAVI has not been reported.

In this paper, DAVI is introduced to reduce the vibration of an underwater vehicle caused by the propeller forces at the propeller fundamental resonance frequency. A semi-analytical dynamic model of the propeller-shaft-hull system with DAVI is set up using the transfer matrix method and substructure synthesis method, in which the elastic properties of the propeller and foundation are taken into consideration. With the efficiency of the proposed semi-analytical model being verified by FEM, the force transmissibility is calculated to evaluate the performance of DAVI. Then, parametric studies are conducted to reveal the effects of the parameters of DAVI on the vibration control performance.

2. Semi-analytical modelling of the underwater vehicle

The simplified schematic of the propeller-shaft-hull system without any isolator is shown in Fig. 1(a), in which only the axial excitation is considered. As shown in this figure, the propeller-shaft-hull system is divided into four subsystems to enable a modular description of the structure. A three bladed propeller is attached to a continuous model of the shaft with length l_s, cross-sectional area A_s, Young's modulus E_s and density ρ_s. l_{se} represents the distance between the propeller and the thrust bearing. The thrust bearing is modelled using a mass-spring-damper system denoted by m_b, k_b and c_b, respectively. The motor and its isolator are simplified as lumped mass m_R and lumped stiffness K_R, respectively. The foundation and hull are treated as impedance boundary conditions. The displacement of the propeller, the thrust bearing, the foundation are denoted by x_p, x, y, respectively, while the corresponding forces are f_p, f_x, and f_y, respectively.

As mentioned DAVI is used to suppress the axial vibration of the underwater vehicle shafting system. It's schematic diagrams of the isolator is shown in Fig. 1 (b). The DAVI is characterized by isolator mass m_{is}, damping coefficient c_b and a lever with length retia of arm $\alpha = l_1/l_2$.

In the following, the dynamic characteristics of the elastic propeller and elastic foundation, including the hull, will be given in terms of receptances using FEM. Then the dynamic model of the DAVI will be deduced based on the lagrangian method. Finally, a semi-analytical model of the propeller-shaft-hull system is to be set up using the substructure synthesis method and the transfer matrix method.

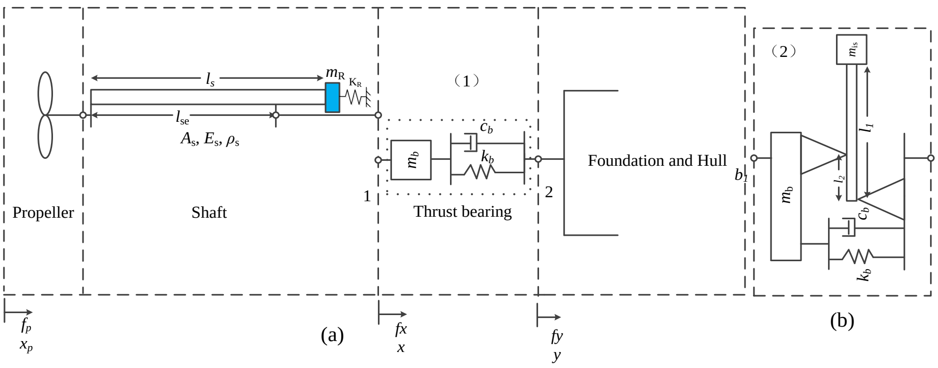

Fig. 1 Simplified model of the propeller-shaft-hull system with DAVI

The propeller and the foundation are termed as the elastic boundary conditions of the shafting system. The receptances of interest are calculated based on FEM model in which the coupling of the fluid/structure are taken into consideration.

A three bladed propeller and a foundation with hull are shown in Fig. 2 , respectively. The surrounding water, which is not shown in the figures for the sake of clarity, is also modelled using FEM.

There are three points in Fig. 2 (a) and (b). Point 1 denotes the junction point at the interface of the propeller hub and the shaft. Point 2 denotes the geometry centre of blade. Point 3 stands for the junction point at the interface of the thrust bearing and the foundation. The cross receptance of $H_{12}(\omega)$ and driving point receptances of $H_{11}(\omega)$ and $H_{33}(\omega)$ are calculated based on FEM model in which fluid/structure interactions are taken into consideration.

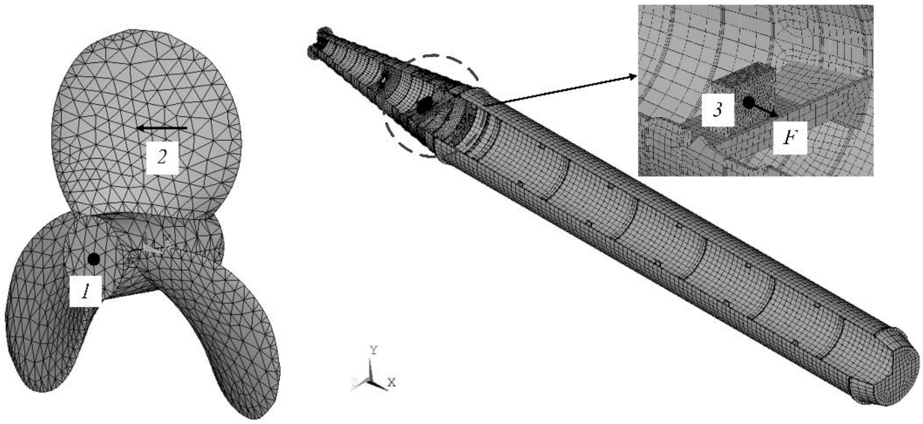

Fig. 2 (a) Finite element model of the propeller and (b) the hull of the underwater vehicle (the fluid field is not displayed)

2.1 Dynamic characteristics of the isolators

Before giving the kinetic equation of the DAVI, let's make two assumptions: (1) The mass m_b is rigidly connected with the massless lever; (2) The damping and the friction in the hinge is negligible;

By kinematic analysis, the kinetic energy, the potential energy and the dissipation function of the DAVI are deduced as

$$T=\frac{1}{2}m_b\dot{x}^2+\frac{1}{2}m_{is}\dot{z}^2,\ \ V=\frac{1}{2}k_b(x-y)^2,\ \ D=\frac{1}{2}c_b(\dot{x}-\dot{y})^2 . \tag{1}$$

The Lagrangian function of the DAVI is

$$L=T-V=\frac{1}{2}m_b\dot{x}^2+\frac{1}{2}m_{is}[(1-\alpha)\dot{y}+\alpha\dot{x}]^2-\frac{1}{2}k_b(x-y)^2 . \tag{2}$$

where $z=(1-\alpha)y+\alpha x$ is the displacement of the isolator mass.

For the variable of x and y

$$\begin{cases}\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}})-\frac{\partial L}{\partial x}+\frac{\partial D}{\partial \dot{x}}=(m_b+m_{is}\alpha^2)\ddot{x}+m_{is}\alpha(1-\alpha)\ddot{y}+c_b(\dot{x}-\dot{y})+k_b(x-y)=F_x \\ \frac{d}{dt}(\frac{\partial L}{\partial \dot{y}})-\frac{\partial L}{\partial y}+\frac{\partial D}{\partial \dot{y}}=m_{is}\alpha(1-\alpha)\ddot{x}+m_{is}(1-\alpha)^2\ddot{y}-c_b(\dot{x}-\dot{y})-k_b(x-y)=F_y\end{cases} . \tag{3 a-b}$$

Assuming

$$x=X\mathrm{e}^{\mathrm{i}\omega t},\ \ y=Y\mathrm{e}^{\mathrm{i}\omega t} , \tag{4}$$

where X, Y are the magnitude of the vibration of the mass m_b and the foundation, respectively.

According to Fig. 1(a), Y and F_y must satisfy the relation

$$\frac{Y}{F_y}=H_{33}(\omega) . \tag{5}$$

Substituting Eq. (4) and Eq. (5) into Eq. (3 a-b),yields

$$Y = \frac{[k_b + \mathrm{i}c_b\omega + \omega^2 m_{is}\alpha(1-\alpha)]H_{33}}{[k_b + \mathrm{i}c_b\omega - \omega^2 m_{is}(1-\alpha)^2]H_{33} - 1} X \ . \qquad (6)$$

Let the variables m_{is} and α in Eq. (6) be zero, the expression of Y_o can be obtained.

2.2 Force and power transmission through the propeller-shafting system

Substituting Eq. (6) into Eq. (5), the magnitude of foundation reaction force under the excitation of the propeller for the system with DAVI can be written as:

$$F_{y,DAVI} = \frac{[k_b + ic_b\omega + \omega^2 m_{is}\alpha(1-\alpha)]}{[k_b + ic_b\omega - \omega^2 m_{is}(1-\alpha)^2]H_{33} - 1} X \ . \qquad (7)$$

Equating the numerator of Eq. (7) to zero, yields the undamped isolation frequency:

$$\omega_{z,DAVI} = \sqrt{\frac{k}{m_{is}\alpha(\alpha-1)}} \ . \qquad (8)$$

Letting the variables m_{is} and α in Eq. (7) be zero, the magnitude of foundation reaction force $F_{y,o}$ when there is without any isolator can be obtained.

The force transmissibility T_F is defined as

$$T_F = 20\log_{10}(F_{hi}/F_p) \ . \qquad (9)$$

where F_{hi} can be $F_{y,o}$, $F_{y,DAVI}$ for the cases of the system without any isolator or with DAVI.

2.3 Dynamic model of the propeller-shaft-hull system

Based upon the results obtained in section 2.1 and section 2.2, the substructure synthesis method is applied to model the propeller-shaft-hull system. As illustrated in Fig. 1(a), the propeller-shaft-hull system is modelled using a rod with an elastic propeller on the left, a motor on the right and one discontinuity at the thrust bearing. Three assumptions are adopted: (a) the material of the rod is linear elastic; (b) the cross sectional of the rod is symmetrical; and (c) the actual dimensions of the rod is assumed to be negligible, and the discontinuities are treated as points connecting the neutral axis of the rod.

According to Fig. 1 (a), the shaft can be divided into two parts. Longitudinal vibration of any part i (i=1,2) can be described independently by $u_i(x_i, t)$, where x_i is the associated local coordinate. The length of each segment is l_{se} and l_s-l_{se}. For each part, the governing equation of the free longitudinal vibration is expressed as

$$E_s A_s \frac{\partial^2 u_i(x_i,t)}{\partial x_i^2} - \rho_s A_s \frac{\partial^2 u_i(x_i,t)}{\partial t^2} = 0 \ , (i=1,2). \qquad (10)$$

where $u_i(x_i,t)$ is the longitudinal displacement of the i-th part at position x_i and time t.

To solve the homogeneous problem, the separation of variables of Eq.(10) is assumed

$$u_i(x_i,t) = U_i(x_i)e^{j\omega t} = (C_i \cos\lambda x_i + D_i \sin\lambda x_i)e^{j\omega t} \ , \qquad (11)$$

where ω is the angular frequency. $U_i(x_i)$ is the mode shape function of the i-th part of the shaft $\lambda=\omega/c$ is the longitudinal wavenumber. $c=\sqrt{E_s/\rho_s}$ is the longitudinal wave speed of the shaft, C_i and D_i are the integration constants determined by the boundary conditions.

The left boundary condition is an elastic propeller. Using the substructure synthesis method introduced by [23], the displacement at the common interface of the propeller and shaft satisfies

$$\mathbf{B}_L \begin{bmatrix} C_1 \\ D_1 \end{bmatrix} = [-1 \quad E_s A_s \lambda H_{11}(\omega)] \begin{bmatrix} C_1 \\ D_1 \end{bmatrix} = -F_e H_{12}(\omega) \ , \qquad (12)$$

where F_p is the magnitude of f_p.

The right boundary condition is a concentrated mass. According to the force continuity condition, its dynamic equation can be given as

$$E_s A_s \frac{\partial u_2(l_s - l_{se}, t)}{\partial x} = -K_R u_2(l_s - l_{se}, t) - m_R \frac{\partial u_2^2(l_s - l_{se}, t)}{\partial t^2}, \quad (13)$$

For the discontinuities condition caused by the thrust bearing, considering the compatibility conditions across the discontinuity point for the continuous of displacement and the interaction load, the following formulae can be deduced

$$\begin{cases} u_2(0,t) = u_1(l_{se}, t) \\ E_s A_s \dfrac{\partial u_2(0,t)}{\partial x} = E_s A_s \dfrac{\partial u_1(l_{se}, t)}{\partial x} + \bar{K}_{1,o} u_1(l_{se}, t) \end{cases}, \quad (14)$$

where $\bar{K}_{1,o} = \{(k_b + \mathrm{i}c_b\omega) - \omega^2 m_b[1 + H_{33}(k_b + \mathrm{i}c_b\omega)]\} / [1 + H_{33}(k_b + \mathrm{i}c_b\omega)]$, it should be $\bar{K}_{1,\mathrm{DAVI}}$, when the isolator is DAVI. Further, according to the Fig. 1 (a), the displacement relationship between the shaft and thrust bearing should satisfy $u_1(l_{se},t)=X_o e^{i\omega t}$, and it equals to $Xe^{i\omega t}$ when the utilized isolator is DAVI.

Combining Eq. (11), Eq. (13) and Eq. (14), the right boundary condition can be written in a matrix form as

$$0 = \begin{bmatrix} r_{11} & r_{12} \end{bmatrix} \begin{Bmatrix} C_2 \\ D_2 \end{Bmatrix} = \mathbf{R}_{1\times 2} \mathbf{T}_{2\times 2} \begin{Bmatrix} C_1 \\ D_1 \end{Bmatrix} = \mathbf{B}_R \begin{Bmatrix} C_1 \\ D_1 \end{Bmatrix} = 0, \quad (15)$$

where $\mathbf{R}_{1\times 2} = \begin{bmatrix} \tilde{K}_R \cos\lambda l_s - \lambda \sin\lambda l_s & \tilde{K}_R \sin\lambda l_s + \lambda \cos\lambda l_s \end{bmatrix}$ and $\tilde{K}_R = (K_R - m_R\omega^2)/(E_s A_s)$.

Combining Eq. (12) with Eq. (15), the constants in the first part of the shaft can be given as

$$\begin{bmatrix} C_1 \\ D_1 \end{bmatrix} = -\begin{bmatrix} \mathbf{B}_L \\ \mathbf{B}_R \end{bmatrix}^{-1} \begin{bmatrix} F_e H_{12} \\ 0 \end{bmatrix}. \quad (16)$$

The FRF of the over shaft can be written as

$$U(\bar{x}) = \sum_{i=1}^{2} \left[C_i \cos\lambda(\bar{x} - \bar{x}_{i-1}) + D_i \sin\lambda(\bar{x} - \bar{x}_{i-1}) \right] \left[H(\bar{x} - \bar{x}_{i-1}) - H(\bar{x} - \bar{x}_i) \right], \quad (17)$$

where the $\bar{x}$ is the global coordinate, $\bar{x}_i$ is the global coordinate of the *i*-th part of the shaft and $H(\bar{x} - \bar{x}_0)$ is the Heaviside function which jumps from zero to unit at location $\bar{x}_0$ ($\bar{x}_{i-1}$ or $\bar{x}_i$).

3. Numerical results and discussions

In this section, the performance of the DAVI will be discussed through a numerical example. The physical properties of the propeller-shaft-hull are listed in Table 1,Table 2 and Table 3, respectively.

Table 1 Parameters of the propeller-shafting system

Parameter	Symbol	Value
Propeller's density	ρ_p	8900 kg/m^3
Propeller's Young's modulus	E_p	127 Gpa
Propeller's Poisson's ration	υ	0.3
Propeller's Radius	r_p	1.5 m
Length of shaft	l_s	15.755 m
Effective Length of shaft	l_{se}	13.92 m
Mass density of shaft	ρ_s	7800 kg/m^3
Young's modulus of shaft	E_s	210 Gpa
Poisson's ration of shaft	υ	0.3
Thrust bearing stiffness	k_b	1.6×10^8 N/m
Thrust bearing mass	m_b	345.37 kg

Table 2 Parameters of the hull

Parameter	Symbol	Value
Cylinder length	L_h	57.2 m
Cylinder radius	R	4.3 m
Shell thickness	t	0.1 m
Stiffener spacing(cone)	d_t	1.25 m
Stiffener spacing(cylinder)	d_c	9 m
Cross sectional area of rectangular section	A_s	0.0162 m^2
Cross sectional area of T section	A_t	0.0722 m^2
Young's modulus of the hull	E_h	210 Gpa
Poisson's ratio of the structure	υ	0.3
Structure loss factor	η	0.02
Added mass		1560 kg/m^3
Density of structure	ρ_h	7800 kg/m^3
Cone half angle	α_c	9.5°
Cone length	l_c	18.5 m
Cone smaller radius	b_c	1.2 m
Density of fluid	ρ_f	1000 kg/m^3
Speed of sound	c_f	1500 m/s
Radius of fluid	r_f	114 m

According to the parameters given above and the method given in section 2, the fundamental resonance frequency of the shaft and the first natural frequency of the propeller can be obtained, which are 15.8 Hz and 47 Hz. The parameters of the DAVI is given in Table 3.

Table 3 Parameters of the DAVI

Parameter	Symbol	Value
Isolator of the mass	m_{is}	25.48 kg
Lever ratio	α	9
External force	F_e	3 N

3.1 Force Transmissibility

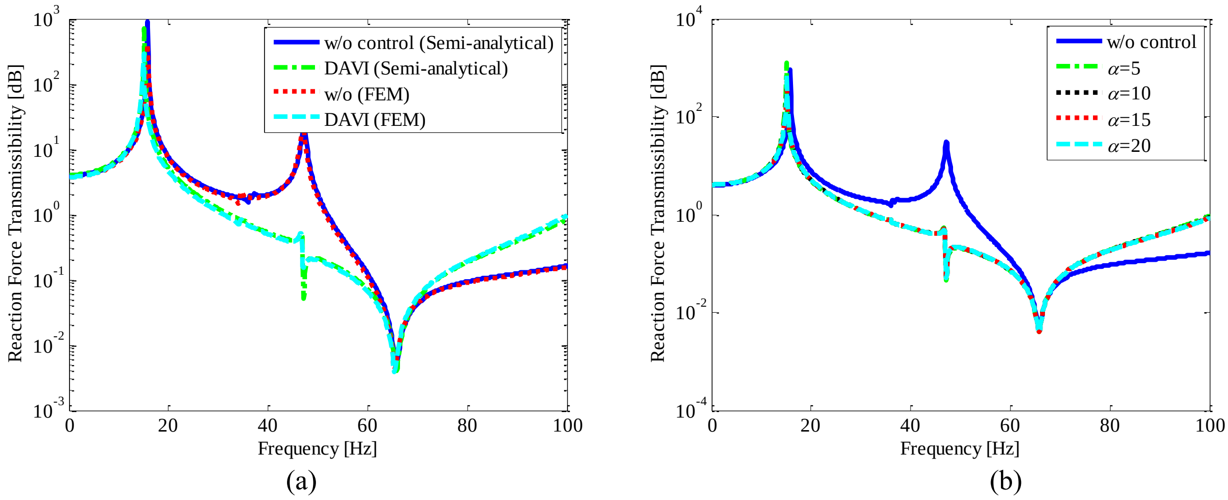

Fig. 3 (a) Reaction force transmissibility from propeller-shafting system to foundation by FEM and semi-analytical model: without any isolator calculated by semi-analytical model (blue solid line), with the DAVI calculated by semi-analytical model (green dash-dot line), without any isolator calculated by FEM (red dot line), with the DAVI calculated by FEM (green dash line) and (b) Sensitivity of the reaction force to α

Fig. 3 (a) compares the force transmissibility from the shafting system to the foundation. For the propeller-shaft-hull system without control, the two peaks which occur at approximately 15.8 Hz and

47 Hz, are attributed to the first axial resonance frequency of the shaft and first natural frequency of the propeller. From this figure, we can see that when the DAVI isolator is applied, the amplitude of the reaction force at the first natural frequency of the propeller is reduced significantly. At the same time the DAVI has few influence on the fundamental resonance frequency of the shafting system. Fig. 3 (a) also gives results of the force transmissibility with and without a DAVI using FEM. It can be seen that the results using semi-analytical method are in a good agreement with the calculations by FEM, which validates the accuracy of the proposed semi-analytical model.

3.2 Sensitivity of the reaction force to α

In this section, the validated semi-analytical method is used to determine the sensitivity of the foundation reaction force to α. When the excitation frequency is equal to Eq.(8), the reaction force of foundation equals to zero.

For a given submarine, both of k_b and the isolation frequency are fixed, so the variables of interests are the m_{is} or α.

The foundation reaction force is calculated with the lever ratio changes from 2 to 20 and the increment is 1. For clarity, only four sets of results are presented in Fig. 3 (b). Generally speaking, the reaction force is not sensitive to leverage ratio and the DAVI have a good isolation performance at the propeller's first natural frequency. Eq. (8) indicates that the smaller the lever ratio is the heavier the isolator mass is and vice versa. Therefore, in reality, a larger lever ratio may be expected which results in a lower weight of the isolator if the installation space for the DAVI is permitted. And in order to realize the larger lever ratio, the hydraulic leverage would be a good choice.

4. Conclusion

In this study, DAVI is firstly used to reduce the vibration transmission through the shafting system of an underwater vehicle, which are induced by the propeller's first vibration mode. A semi-analytical model of the propeller-shaft-hull system is developed, taking the elastic properties of propeller and foundation into consideration. The force transmissibility is utilized to evaluate the isolation performance of the DAVI. The numerical results show that the DAVI can not only effectively isolate the vibration at the designed frequency, but has few effects on the axial fundamental resonance frequency of the shaft. Sensitivity analysis of the reaction force to lever ratio α shows that the lever ratio has negligible effects on the isolation performance. If the installation space of DAVI is permitted, a larger lever is expected, since the weight of the isolator is lower. Future research will concentrate on experiment verification of the efficiency of the DAVI.

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