# THE ANALYSIS OF SIGNALS CONTAINING MIXTURES OF LINEAR CHIRPS

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## 1. ABSTRACT

*This paper presents a method for the analysis of signals containing mixtures of linear chirp signals. Such signals occur frequently in geophysics and in acoustic signals such as bats and dolphins. In a discussion of the method, such matters as chirp rate and time resolution will be dealt with and also the viewing of results. The authors present the material as an extension to Fourier analysis where frequency resolution and filtering are analogous. As well as the derivation of some of the properties of the method, an algorithm is also presented which allows practical analysis of chirp mixture signals. Examples will be presented to illustrate the ideas.*

**KEY WORDS:** chirp signals, Fourier analysis

## 2. INTRODUCTION

In this paper we attempt to formalise a method for the analysis of signals containing one or more linear chirps. Such signals arise in geophysical data from signals obtained by reflections of input chirps into layered media [1], and in bat and dolphin signals [2].

We seek to gain a greater understanding of chirp analysis and describe an algorithm and practical method for the detection of linear chirps in multiple chirp signals. We also address the questions of chirp resolution in rate, start frequency and time, the number of samples to resolve chirps and the rate of sampling required for a given chirp analysis. Chirps are in a class of signals whose frequencies change with time. This enters the domain of time frequency analysis, where conventional Fourier analysis is not appropriate [3].

The chirps in the signal may occur at different start times and they may have differing parameters. In the discussion it is hoped to answer such relevant questions as:

- What is the rate resolution between two chirps,
- What is the time resolution between two chirps, and
- Can certain chirps be filtered from the data?

In other words, can we begin to answer similar questions that are posed (and answered) by the Fourier transform acting on sinusoidal signals.

## 3. ANALYSIS METHODOLOGY

Conventional Fourier analysis is concerned with signals containing mixtures of sinusoids. Such signals may be represented as:

$$g(t) = \sum_k \exp(ja_k t) \qquad (1)$$

We note that these *basis* functions have linear phase. Given that the instantaneous frequency is the derivative of the phase, in the above case it is a zero order function; constant over time. The analysis of such signals involves eliminating each frequency in turn through multiplying by the complex conjugate of the sinusoid at the particular frequency. This effectively *translates* that frequency to the zero frequency (dc), therefore isolating that term. The analysis proceeds thus for each frequency in turn.

The general complex linear chirp is given by the following expression:

$$f(t) = A\exp(j(at^2/2 + bt + c)) \tag{2}$$

The instantaneous frequency ($f_I$) for equation 2 is related to time as $f_I(t) = at + b$. In the time frequency plane this is a first order function, allowing the frequency to vary with time. The chirp in equation 2 may be recognised as having the following properties:

- A rate $a$ Hz/sec,
- A start frequency $b$ Hz, and
- An initial phase $c$ radians.

Consider multiplication of this general chirp by a unit-amplitude chirp with zero start frequency, zero initial phase, and rate $a'$. The result produces the following:

$$f(t) = A\exp(j((a + a')t^2/2 + bt + c)) \tag{3}$$

Note that when $a' = -a$, the result is

$$f(t) = A\exp(j(bt + c)) \tag{4}$$

which is a single sinusoid of frequency $b$. If this signal is now Fourier transformed, we may obtain the parameters $A$, $b$, and $c$. In effect, given that we have found chirp rate $a$, we have all the parameters of the original chirp signal. The chirp rate may be found by sweeping through the range of values in exactly the same manner as a Fourier transform sweeps through the range of frequency values. At some particular rate $a'$, which matches $a$, we remove the chirp and isolate the residual frequency component. The algorithm is thus:

> *Over a range of chirp rates, for each rate in turn, multiply the signal being analysed by this chirp, and Fourier transform the result.*

Since we are now evaluating two variables, $a$ and $b$, a convenient way to view the result of the analysis is as an image. This leads to a two-dimensional display of chirp rate against residual frequency, where two displays may be obtained, one for magnitude and one for phase.

In order to gain some understanding of the method and the problems involved let us consider the chirp described by the parameters shown in table 1, with total sample time $T = 2.0$ seconds and $N = 512$ samples; a Gaussian envelope was used in the signals. The chirp start and end times and frequencies are $t_1$, $t_2$, $f_1$, and $f_2$ respectively. We define $f_s$ as the sampling frequency in Hz, and the chirp rate $a$ as,

$$a = \frac{f_2 - f_1}{t_2 - t_1} \tag{5}$$

| amplitude | $t_1$ (seconds) | $t_2$ (seconds) | $f_1$ (Hz) | $f_2$ (Hz) |
|---|---|---|---|---|
| 100.0 | 0.0 | 2.0 | 10.0 | 20.0 |
| 100.0 | 0.0 | 2.0 | 30.0 | 55.0 |
| 100.0 | 0.0 | 2.0 | 60.0 | 95.0 |
| 100.0 | 0.0 | 2.0 | 80.0 | 125.0 |

Table 1: Chirp parameters for Fourier extension example.

If this signal is analysed by the Fourier extension method we produce the left hand image shown in figure 1.

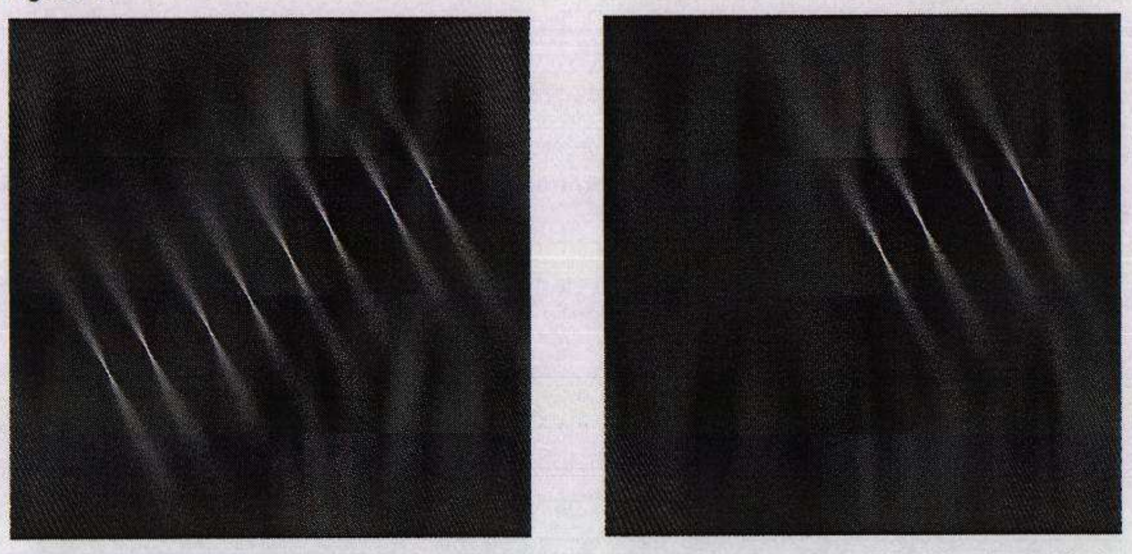

Figure 1: Fourier Extension Images of the real (left), and analytic (right) signals, defined in table 1.

In the left image we notice two distinct focal points corresponding to each chirp in this signal. These correspond to one for the positive frequency and one for the negative frequency (the input was a real signal only, not complex). The analytic signal may be produced using the Hilbert transform of the real signal [4], and this produces positive frequencies only, as shown in the right hand image of figure 1.

In our image the focal point lines occur at positions $\pm aT^2$ about the centre line. This will be derived later when we consider the resolution of chirp rates. To derive the relationship between chirp parameters and position of the residual frequency peak, we consider the chirp beginning at time $t_1$:

$$f(t) = A\exp(j(a(t-t_1)^2/2 + b(t-t_1) + c) \tag{6}$$

This corresponds to instantaneous frequency $f_i = a(t-t_1)+b$.

At $t = 0$ the instantaneous frequency is $b - at_1$. It is this frequency, which appears as a residual frequency when the signal is multiplied by the optimal analysing chirp.

Hence the position of the optimal peak (residual frequency) is given by,

$$\frac{b-at_1}{(f_s/N)}=(b-at_1)T \qquad (7)$$

Just as with Fourier analysis, the frequencies, which may be found, are dependent on the number of samples (and the rate at which the samples are obtained). Similarly in this Fourier extension analysis, the chirps, which may be found, are also dependent on the number of samples in the signal.

Now we have some basic relationships between chirp parameters and the occurrence of the optimal peak in the rate-"residual frequency" image, we can turn our attention to the relationships concerning separation, or resolution, between linear chirps.

## 4. THE RESOLUTION OF CHIRPS IN RATE AND TIME

In this section we consider the resolution between two chirps in both rate and time. This is analogous to Fourier analysis where frequency resolution is dependent on $1/T$.

### 4.1 Time Resolution Between Two Linear Chirps

We consider two chirps of the same rate with start positions separated by *dt* seconds. If they were of different rate we would also have separation in the rate direction. Here we consider the case of identical chirp rates. Let the chirps be

$$\begin{aligned} f_1(t)&=\exp(j(\frac{a}{2}t^2)),\\ f_2(t)&=\exp(j(\frac{a}{2}(t+dt)^2)) \end{aligned} \qquad (8)$$

i.e. linear chirps with instantaneous frequencies $f_{I_1}=at$ and $f_{I_2}=a(t+dt)$ respectively, separated by $a.dt$ in frequency at $t=0$. Let the analysing chirp be $\exp(-jat^2/2)$. Multiplying the signal consisting of two linear chirps by this analysing chirp produces two components, one at dc, corresponding to chirp 1, and one at $a.dt$ corresponding to chirp 2.

The frequency resolution is $f_s/N$, for separation it is required that $a.dt=f_s/N$, therefore the chirp time resolution ( $CTR$ ) is,

$$CTR=dt=\frac{f_s}{aN}=\frac{1}{aT} \qquad (9)$$

This will result in two residual frequency peaks being in two adjacent frequency bins. If we require them to be separated by 2 frequency bins, then $dt=2/aT$. As an example consider the separation of two chirps with parameters as shown in table 2; each chirp has a rate of 1Hz/sec and starts at 0Hz, with $T=10$ seconds, and $N=1024$ samples.

| $a$ (Hz/sec) | $t_1$ (seconds) | $t_2$ (seconds) | $f_1$ (Hz) | $f_2$ (Hz) |
|---|---|---|---|---|
| 1.0 | 0.0 | 9.0 | 0.0 | 9.0 |
| 1.0 | 0.2 | 9.2 | 0.0 | 9.0 |

Table 2: Chirp parameters for Rate separation experiment.

From equation 9 the resolution is 0.1 seconds, this example has a separation of 0.2 seconds, or two frequency bins. This can also be derived from equation 7 which for chirp 1 gives a residual frequency at position zero, and for chirp 2 gives a residual frequency bin at position –2. This is illustrated in figure 2, where an expanded view of the residual frequency axis is plotted; note the peaks occur at 512+0=512 and 512-2=510, where 512 is the dc position.

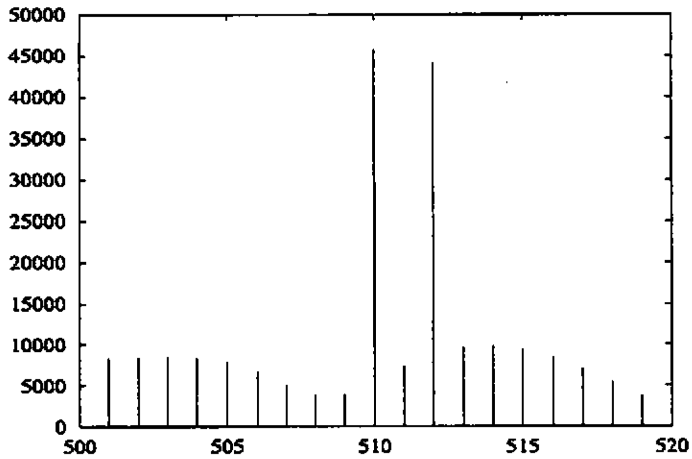


Figure 2: Time separation between two chirps in table 2.

### 4.2 The Resolution Between Two Linear Chirps

The fastest chirp occupying the total length of the signal that can be represented without aliasing has a slope $a=(f_s/2)/T$. The slowest chirp occupying the total length of the signal (apart from a zero chirp, i.e. sinusoid) corresponds to the rate $(f_s/N)/T=f_s/(NT)$. The other chirps can then be considered as part of the set with rate given as $a_k=kf_s/(NT)$, where $k=1,\ldots,N/2$.

In theory two chirps can be separated when their rates differ by $da=f_s/(NT)$ (i.e. when the optimal transform for each chirp occurs in adjacent rate bins), therefore the chirp rate resolution ($CRR$) is

$$CRR=da=\frac{f_s}{NT}=\frac{1}{T^2} \qquad (10)$$

(Note here the analogy to Fourier analysis of sinusoids where the frequency resolution is $1/T$).

As an example, consider the separation of the following two chirps whose parameters are given in table 3; again for $T = 10$ seconds, and $N = 1024$ samples.

| $a$ (Hz/sec) | $t_1$ (seconds) | $t_2$ (seconds) | $f_1$ (Hz) | $f_2$ (Hz) |
|---|---|---|---|---|
| 0.5 | 0.0 | 10.0 | 0.0 | 5.0 |
| 0.51 | 0.0 | 10.0 | 0.0 | 5.1 |

Table 3: Chirp parameters for Rate separation experiment.

In this example $da = 0.01$. If $da < 0.01$ we could not resolve the peaks. The result of the analysis is shown in figure 3, which is a plot of amplitude against normalised rate ($aT^2$).

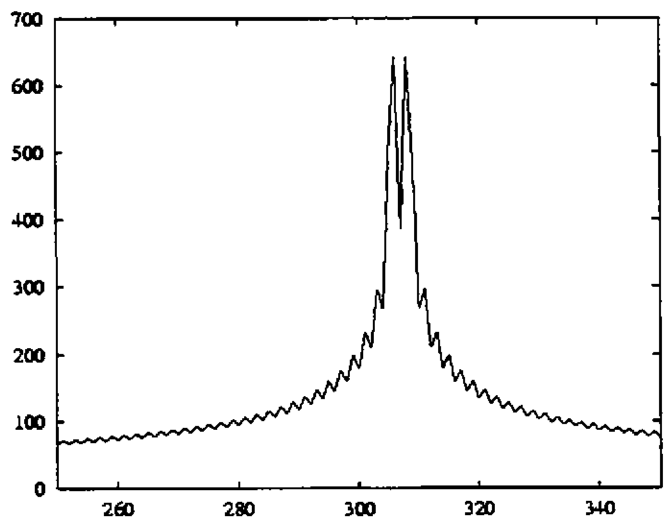


Figure 3: Rate separation between two chirps in table 3.

The two maximum peaks are indeed in adjacent rate bins, however this does not preclude the possibility of a single chirp with a rate between the two chirps. That is, between 0.5 and 0.51. This is similar to the situation in Fourier analysis where we cannot find a frequency which is not an integer multiple of the frequency resolution.

We now consider a practical algorithm for the analysis of mixtures of linear chirp signals.

## 5. AN ALGORITHM FOR THE ANALYSIS OF LINEAR CHIRP SIGNALS

Our first step in analysing a signal was multiplication by a chirp function and then Fourier transformation. Considering figure 4 we can gain some insight into the mechanism to be adopted for analysis. This will also aid in the understanding of the resolution aspects.

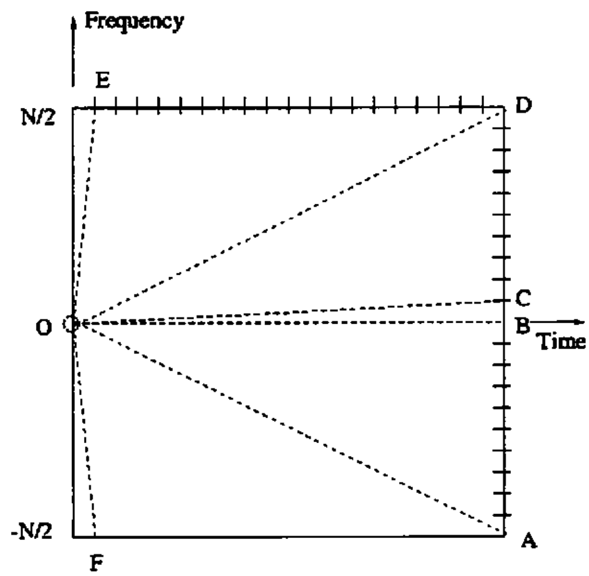


Figure 4: Time-Frequency Sampling.

This figure shows a representation of the time-frequency domain. In time we have N samples equivalent to a total sampling time of *T* seconds (and thus a sampling rate of $f_s = N/T$). We can consider the frequency scale to go from $-N/2...N/2$. A pure sinusoid (zero rate chirp) would be represented on the graph by the line OB. The slowest positive rate chirp (apart from zero rate) would be represented by the line OC. The fastest positive rate chirp occupying the total time of the signal would be represented by the line OD. The corresponding negative rate chirp would be represented by OA.

Let us consider what these different rate chirps correspond to in terms of frequencies and times. For the chirp corresponding to OD. This is the fastest positive rate chirp that occupies the total sample time of the signal without aliasing. Its maximum frequency is $f_s/2 = N/(2T)$. This occurs in *T* seconds, so the rate *a* is $(N/(2T))/T = N/(2T^2)$. Similarly, the negative chirp OA has rate $-N/(2T^2)$. Thus the slowest positive rate chirp OC (neglecting the zero rate OB) has rate $(1/T)/T = 1/T^2$. Given these facts we observe that, given this sampling system:

- The chirp rate resolution is linear for rates up to $N/(2T^2)$ and equal to $1/T^2$
- The maximum rate of chirp occupying the total length of the signal is $N/(2T^2)$
- To increase the maximum rate of chirp we must increase $f_s$ for the same total sample time *T*.
- The chirp time resolution is linear for rates up to $N/(2T^2)$ and equal to $1/aT$.
- To increase the resolution we must increase the total sample time *T* for the same sampling rate $f_s$.

Given such a system we can devise an algorithm as follows:

- for *a* = $-N/(2T^2)$ to $N/(2T^2)$ step $1/T^2$

- Multiply signal by this chirp
- Forward Fourier transform this result

This means that for *N* samples we generate $N+1$ possible chirps (NB. this is not all the discrete possible chirps). The first chirp in this progression is chirp OA. As we proceed towards chirp OD, the $i$ th chirp in this progression, where $i=0,\ldots,N$, will have a rate

$$\frac{-((N-i)/2T)}{N.dt}=\frac{-((N-i)/2)}{T^2} \tag{11}$$

where $dt=T/N$, giving $N+1$ chirps.

Each transformed signal will have length *N* following the Fourier Transform and so we can imagine all this data being represented by an image with $N+1$ rows and $N$ columns. The centre row corresponds to the zero rate chirp or the normal Fourier transform. The top row of the image corresponds to the fastest negative rate chirp, and the bottom row of the image to the fastest positive rate chirp. Remember, the first chirp we multiply by is the most negative rate and this takes out the most positive rate chirp if it is present in the signal. Each row of the image represents a chirp of a particular rate and each row is separated by the chirp resolution of $1/T^2$.

## 6. CONCLUSIONS

This paper has considered the factors affecting the separation of mixtures of linear chirps. Several basic relationships have been derived, addressing particularly the problems of separation of chirps in rate and time. An algorithm was also presented whereby practical chirp signals could be analysed, and a method of visualising the result was given. Some applications of the algorithm will be presented during the conference.

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