ON SOUND TRANSMISSION ACROSS AN EXPONENTIAL BOUNDARY LAYER

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ABSTRACT

We consider (§1) an acoustic wave of frequency ω and horizontal wavenumber k in an exponential boundary layer of free stream velocity V and thickness scale L. It is shown that a critical layer exists in the boundary layer if $\omega<kV$, and that it acts as an 'acoustic valve' (§2). Both the critical layer (§4) and the free stream (§3) are regular singularities (§6) of the wave equation. In their neighbourhood exist ascending power series solutions, specifying propagating, evanescent or divergent waves in the free stream (§3); the acoustic field can have a logarithmic singularity at the critical layer (§4). It is possible to perform analytic continuation (§5) between the solutions in the neighbourhood of the three singularities of the wave equation, two regular and one irregular. The irregular singularity corresponds to the limit of infinitely strong vorticity below the wall (§7), and the general solution in its vicinity involves infinite determinants, as for Hill's equation (§8), although a particular solution exists as a normal integral (§9). Among the many possible cases of possible interest (§10) we plot the acoustic pressure in a supersonic boundary layer with a critical level.

1. INTRODUCTION

The transmission of sound [1] across boundary layer is important in at least two contexts: (i) the attenuation of the noise emitted by an engine and received in aircraft cabin; (ii) the acoustic propagation in the atmospheric layers near the ground. We take as an example an uniform stream in the x-direction sheared in the y-direction, i.e. with velocity $\vec{v}=U(y)\,\vec{e}_x$. An acoustic wave of frequency ω and wavenumber k in the x-direction has an acoustic pressure P, whose dependence on y is specified [2] by the differential equation:

$$P'' + \{2kU'/(\omega-kU)\}\,P' + [(\omega-kU)^2/L^2-k^2]\,P=0, \quad (1)$$

where prime denotes derivative with regard to y, e.g. $U'=dU/dy$. The factor in curly brackets is the Doppler shifted frequency:

$$\omega_*(y)=\omega-k\,U(y) = \omega-k\,V(1-e^{-y/L}), \quad (2)$$

which is given by (2) in the case of an exponential boundary layer.

2. CRITICAL LAYER AND VALVE EFFECT

The wave equation (1) has a singularity where the Doppler shifted frequency vanishes, and this $\omega_*(y_c)=0$ specifies the location of the critical layer; in the case (2) of the exponential velocity profile, the critical level

$$y_c=-L\log(1-\Omega), \qquad \Omega \equiv \omega/kU, \tag{3a,b}$$

if the Doppler shifted frequency is negative in the free stream ω-kU<0, in which case it must vanish at an intermediate distance $y=y_c$ from the wall; conversely, if the Doppler shifted frequency is positive in the free stream ω-kU>0, it is positive everywhere, and no critical layer exists, viz. y_c is not real. The critical layer $y=y_c$ is placed at $\zeta=1$ by the change of independent variable:

$$\zeta \equiv e^{-y/L}/(1-\Omega), \qquad P(y;k,\omega)=f(\zeta), \tag{4a,b}$$

for which the wave equation (1) has polynomial coefficients:

$$(1-\zeta)\,\zeta^2 f'' + \zeta(1+\zeta)\, f' + (1-\zeta)\,\{\Lambda^2(1-\zeta)^2 - K^2\}\, f=0, \tag{5}$$

and has solution in series of powers of ζ. In (5) we introduce the dimensionless Doppler shifted frequency and wavenumber:

$$\Lambda=(\omega-kV)\,L/C, \qquad K=kL. \tag{6a,b}$$

The absence of a critical layer $\Omega>1$, the variable is negative $\zeta<0$, the power series has alternating sign, hence a modest sum, and changes of acoustic pressure accross the boundary layer are small. Conversely, in the presence of a critical layer $\Omega<1$, we have a series with fixed sign $\zeta>0$ and large sum, and so there are significant changes of acoustic pressure accross the boundary layer, implying that: (i) a sound wave coming from the free stream is significantly attenuated as it approaches the wall; (ii) a sound wave from the wall is amplified towards the free stream. Thus the critical layer [3], when it exists, acts as an acoustic *'valve'*, favouring propagation in one direction.

3. EVANESCENT, DIVERGENT AND PROPAGATING WAVES

If we perform the change of dependent variable:

$$\Phi(\zeta) = \zeta^{\upsilon} f(\zeta), \qquad \upsilon = \sqrt{\Lambda^2-K^2}, \tag{7a,b}$$

the coefficients of the differential equation became quadratic instead of cubic polynomials:

$$(1-\zeta)\,\zeta\,\Phi''+ \{(1+2\upsilon) + \zeta(1-2\upsilon)\}\,\Phi' + \{2\upsilon-\Lambda^2(1-\zeta)(2-\zeta)\}\,\Phi=0; \tag{8}$$

the free stream $y=\infty$, $\zeta=0$ is a regular singularity [4] of the differential equation (8), and a solution exists as an ascending power series:

$$\Phi_\sigma(\zeta) = \zeta^\sigma \sum_{n=0}^{\infty} C_n \zeta^n, \quad \sigma=0, -2\upsilon, \tag{8a,b}$$

where the index σ may take two values, and the coefficients satisfy the recurrence formula:

$$(n+\sigma+1)(n+\sigma+1+2\upsilon)\, C_{n+1}=$$

$$=\{(n+\sigma)((n+\sigma-2-2\upsilon) + 2(\Lambda^2-\upsilon)\}\, C_n+\Lambda^2(C_{n-2}-3C_{n-1}). \tag{8c}$$

The acoustic field is a linear combination of the particular integrals (8a,b,c), viz.:

$$P(y;k;\omega) = C_+ \zeta^\upsilon \Phi_0(\zeta) + C_- \zeta^\upsilon \Phi_{-2\upsilon}(\zeta), \tag{9}$$

where the $C_\pm$ are arbitrary constants of integration.The two terms of (9) scale as:

$$\zeta^\upsilon\{\Phi_0(\zeta), \Phi_{-2\upsilon}(\zeta)\} \sim \zeta^{\pm\upsilon} \sim e^{\pm\upsilon y/L}, \tag{10}$$

and hence represent: (i) an evanescent and a divergent wave for real υ; (ii) inward and outward propagating waves for imaginary υ. It follows from (7b) that we have propagating waves in the free stream if $K^2>\Lambda^2$ in (6a,b), i.e. if $-kc<\omega-kV<kc$; otherwise, if $\omega>k(V+c)$ or $\omega<k(V-c)$ the acoustic field consists of evanescent and divergent components.

4. ASCENDING POWER SERIES AND LOGARITHMIC SOLUTIONS

The solution (9) is valid for $|\zeta|<1$, which includes the whole flow region $0<y<\infty$ in (4a) if $\Omega>2$. If $1<\Omega<2$ then the solution (9) converges only beyond the layer y_1 given by:

$$y > y_1 = -L \log(\Omega-1), \tag{11}$$

and if $0<\Omega<1$ it converges beyond the critical layer $y>y_c$ in (3a). The latter corresponds to $\zeta=1$, which is a regular singularity of the differential equation (8); hence it has a solution of the form:

$$\Phi^\sigma(\zeta) = \sum_{n=0}^{\infty} b_n (\zeta-1)^{n+\sigma}, \tag{12}$$

where the coefficients satisfy the recurrence formula:

$$(n+\sigma+1)(n+\sigma-2)\, b_{n+1} = \{(n+\sigma)((n+\sigma-2\upsilon)-2\upsilon\}\, b_n + \Lambda^2(b_{n-1}+b_{n-2}); \tag{13}$$

since the index takes the values $\sigma=0,3$ differing by an integer, the linearly independent particular integrals [4] are:

$$\overset{3}{\Phi}(\zeta) \sim (\zeta-1)^3, \quad \overset{-3}{\Phi}(\zeta) \equiv \lim_{\sigma \to 3} \partial \overset{\sigma}{\Phi}(\zeta)/\partial\sigma \sim (\zeta-1)^3 \log(\zeta-1), \tag{14a,b}$$

i.e., one vanishes at the critical layer, and the other has a logarithmic singularity there. Their linear combination specifies the acoustic field in the vicinity of the critical layer:

$$P(y;k,\omega) = C_1 \zeta^{\upsilon} \overset{3}{\Phi}(\zeta) + C_2 \zeta^{\upsilon} \overset{-3}{\Phi}(\zeta), \tag{15}$$

which converges for $|\zeta-1|<1$.

5. THREE REGIONS AND ANALYTIC CONTINUATION

The differential equation has singularities at $\zeta=0,1,\infty$; in the neighbourhood of each singularity there exist two linearly independent particular integrals, for a total of six. They are specified by series of powers; (§3) of ζ, converging for $|\zeta|<1$, about the free stream; (§4) of $\zeta-1$, converging for $|\zeta-1|<1$, about the critical layer; (§5) of $1/\zeta$, converging for $|\zeta|>1$, about infinity below the wall. Since the region $|\zeta-1|<1$ overlaps with $|\zeta|<1$ and $|\zeta|>1$, it is possible to perform analytic continuation, i.e., any particular integral is a linear combination of any two of the other pairs. This scheme is similar to that of hypergeometric equation [5], which has regular singularities at $\zeta=0,1,\infty$, with the important difference that the equation (8) is of higher level of complexity: it would reduce to the hypergeometric type if $\Lambda=0$, but for $\Lambda\neq0$ the singularity at $\zeta=\infty$ is irregular, and reduction to the hypergeometric type is not possible. We proceed to establish the nature of the singularities (regular or irregular) of the differential equation, which determines the type of power series (ascending or ascending-descending) solution in its neighbourhood.

6. TWO REGULAR AND ONE IRREGULAR SINGULARITIES

The nature of the singularity (regular or irregular) of the differential equation, determines the type of power series (ascending or ascending-descending) solution in its neighbourhood. The singularity at the free stream ($y=\infty$, $\zeta=0$) is regular because if the differential equation (8) is put into the form:

$$\zeta^2 \Phi'' + \zeta\, p(\zeta)\, \Phi' + q(\zeta)\, \Phi=0, \tag{16}$$

the functions $p,q\ (\zeta)$ are analytic at $\zeta=0$; hence an ascending power series solution (8a,b,c) exists. Similarly for the regular singularity at the critical layer ($y=y_c$, $\zeta=1$), with the coincidence of exponents leading to a logarithmic singularity in the solution. In order to investigate the singularity of the differential equation (8) in the limit of strong vorticity below the wall ($y=-\infty$, $\zeta=\infty$), we perform an inversion:

$$\eta=1/\zeta,\quad \psi(\eta)=\Phi(\zeta), \tag{17a,b}$$

and consider the singularity at the origin of:

$$(1-\eta)\eta\ \psi'' + (3-\eta)\psi' + \{K^2-\Lambda^2 + (3\Lambda^2-K^2)/\eta - 3\Lambda^2/\eta^2 + \Lambda^2/\eta^3\}\ \psi=0 \tag{18}$$

If the equation is written in the form:

$$\eta^2\psi'' + \eta\, r(\eta)\,\psi' + s(\eta)\,\psi=0, \tag{19}$$

it is clear that $s(\eta)$ has a double pole at $\eta=0$, so this is an irregular singularity of (18), corresponding to the irregular singularity $\zeta=\infty$ of (8). If a solution of (18) was sought by the Frobenius-Fuchs method, in the form of an ascending power series, it would fail to specify an indicial equation.

7. INFINITE DETERMINANTS AND SYSTEMS OF EQUATIONS

In the neighbourhood of the irregular singularity of the differential equation, the solution has an essential singularity, specified [6] by a Laurent series, involving ascending and descending powers:

$$\psi(\zeta) = \sum_{n=-\infty}^{+\infty} d_n\,\eta^{n+\sigma} = \sum_{n=-\infty}^{+\infty} d_n\,\zeta^{-n-\sigma} = \Phi(\zeta). \tag{20}$$

Substitution of (20) into (18) leads to an infinite system of linear homogeneous equations:

$$\sum_{m=-\infty}^{+\infty} D_{n,m}(\sigma)\,d_m=0, \tag{21}$$

with matrix of coefficients having all terms zero, except for four rows along the principal diagonal:

$$D_{n,n} = (n+\sigma)^2 + \Lambda^2-K^2-1,\qquad D_{n,n+2} = -3\Lambda^2, \tag{22a,b}$$

$$D_{n,n+1} = 3\Lambda^2 - (n+\sigma)(n+\sigma+4) - K^2,\quad D_{n,n+3} = -\Lambda^2. \tag{22c,d}$$

The system (21) has non-trivial solution, i.e. the series (20) does not vanish, iff σ is a root of the infinite determinant $|D_{n,m}(\sigma)| = 0$; this is the indicial equation, which has two roots. For each root the system (21) may be solved for the ratio of the coefficients to d_o:

$$\sum_{m=1}^{\infty} \{D_{n,m} (d_m/d_o) + D_{n,-m} (d_{-m}/d_o)\} = D_{n,o}, \tag{23}$$

i.e. this is an infinite system of linear inhomogeneous equations.

8. TRANSFORMATION TO HILL'S DIFFERENTIAL EQUATION

The methods of evaluation of infinite determinants have been developed in connection with Hill's equation, and we note in passing that the equation (18) can be reduced to the latter form. Since we need the solution of (8) for $|\zeta|>1$, or (18) for $|\eta|<1$, we deform the change of independent variable:

$$\eta=\cos^2\theta, \qquad \psi(\eta) = g(\theta), \tag{24a,b}$$

which leads to:

$$g'' - (4 \cot 2\theta + 8 + 2 \tan\theta)\, g' +$$

$$\{K^2-\Lambda^2 + (3\Lambda^2-K^2) \sec^2\theta-3\Lambda^2 \sec^4\theta + \Lambda^2 \sec^6\theta\}\, g=0. \tag{25}$$

The coefficient of g' may be eliminated by the change of dependent variable:

$$g(\theta) = e^{4\theta} \sin\theta\, h(\theta), \quad h'' + J(\theta)\, h=0, \tag{26a,b}$$

leading to a differential equation (26b) with periodic coefficient:

$$J(\theta) \equiv K^2-\Lambda^2 + (3\Lambda^2-K^2+1) \sec^2\theta - 3\Lambda^2 \sec^4\theta +$$

$$+ \Lambda^2 \sec^6\theta - 4 \sec^2 2\theta + (4+\tan\theta+2 \cot 2\theta)^2. \tag{27}$$

The latter can be expanded in series of cosine of even arguments:

$$J(\theta) = \sum_{n=o}^{\infty} A_n \cos(2n\theta), \tag{28}$$

and (26b, 28) specify Hill's equation. The approximations used by Hill [7] do not apply as well here, because the coefficients A_n decay slowly.

9. ESSENTIAL SINGULARITY AND NORMAL INTEGRAL

The general integral in the neighbourhood of the irregular singularity of the differential equation has an essential singularity specified by the preceding methods (§6,7) involving infinite determinants. There exists one particular integral, which can be determined without recourse to infinite systems of equations. It is expressed as a normal integral [4,6] of the form:

$$\psi(\eta) = e^{\Omega(1/\eta)}\, q(\eta), \quad q(\eta) = \sum_{n=o}^{\infty} a_n\, \eta^{n+\sigma}, \tag{29a,b}$$

where the function $\Omega(1/\eta)$ takes care of the essential singularity at $\eta=0$, so that $q(\eta)$ is an ascending power series determined by the Frobenius-Fuchs method. In order to find out whether such a function exists, we substitute (29a) into (18) to obtain a differential equation for q:

$$\eta(\eta-1)\, q'' + \{2\Omega'\eta(\eta-1) + \eta-3\}\, q' + \{(\Omega'^1+\Omega'^2)\, \eta(\eta-1) + (3-\eta)\, \Omega'+\Lambda^2-K^2 +$$

$$+ (k^2-3\Lambda^2)/\eta - 3\Lambda^2/\eta^2 + \Lambda^2/\eta^3\}\, q=0. \tag{30}$$

The latter has a regular singularity at $\eta=0$ if the coefficient of q' is $0(1)$ and the coefficient of q is $0(1/\eta)$. This is met by:

$$\Omega = \pm\, \Lambda/\eta + (5/2 \pm \Lambda) \log \eta, \tag{31}$$

so that the normal integral (29a,b) is given by:

$$\psi(\eta) = e^{-\Lambda/\eta}\, \eta^{5/2-\Lambda} \sum_{n=o}^{\infty} a_n\, \eta^n, \tag{32}$$

where the coefficients a_n satisfy the recurrence formula:

$$2\Lambda(n+\sigma+1)\, a_{n+1} = \{K^2-6\Lambda^2+3\Lambda+15/4-(n+\sigma)(n+\sigma-4\Lambda-7)\}\, a_n$$

$$+ \{(n+\sigma-1)(n+\sigma-6\Lambda+16) + \Lambda^2-K^2\}\, a_{n-1}, \tag{33}$$

which follows from the differential equation for q, viz. (30) with (31):

$$\eta^2(\eta-1)\, q'' + 2\eta\, \{2\Lambda-4-\Lambda/\eta + (3-\Lambda)\, \eta\}\, q' +$$

$$+ \{(\Lambda^2-K^2)\eta + K^2-6\Lambda^2 + 3\Lambda + 15/4\}\, q=0. \tag{34}$$

Note that $\eta=0$ is not a regular singularity of (34), but the Frobenius-Fuchs method leads to an indicial equation of the first degree $\sigma=0$, i.e. we can determine only one particular integral.

10. ACOUSTIC PRESSURE AS A FUNCTION OF DISTANCE FROM THE WALL

The preceding methods specify the acoustic pressure as a function of the distance from the wall, using as necessary the computation [8] of solutions about the three singularities, to cover the whole flow region, for all possible combinations of dimensionless frequency (3a), wavenumber (6b) and Mach number $M=V/c$ of the free stream. Since there are too many combinations of interest, we limit overselves to one example of a boundary layer with a free stream Mach number $M=1.25$, for which the sound field is evanescent; the dimensionless frequency $\Omega=0.2$ is such that a critical layer exists. The acoustic pressure has a dip at the critical layer, and increases: (i) towards the wall, independently of wavenumber; (ii) towards the free stream, faster for larger wavenumber.

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LEGENDS FOR THE FIGURES

FIGURE 1 — Acoustic wave of frequency ω and horizontal wavenumber k in a boundary layer with free stream velocity V and thickness scale L, and exponential velocity and vorticity profiles.

FIGURE 2 — Acoustic pressure normalized to the wall value, as a function od dimensionless distance from the wall $Y \equiv y/L$, for fixed free stream Mach number $M=1.25$ and dimensionless frequency $\Omega \equiv \omega/kV=0.2$, and three values of compactness $K=kL=1,5,10$.

ON SOUND TRANSMISSION ACROSS AN EXPONENTIAL BOUNDARY LAYER

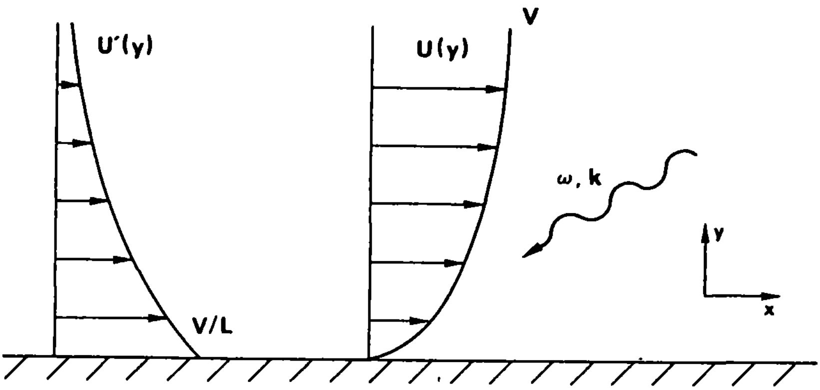

FIGURE 1

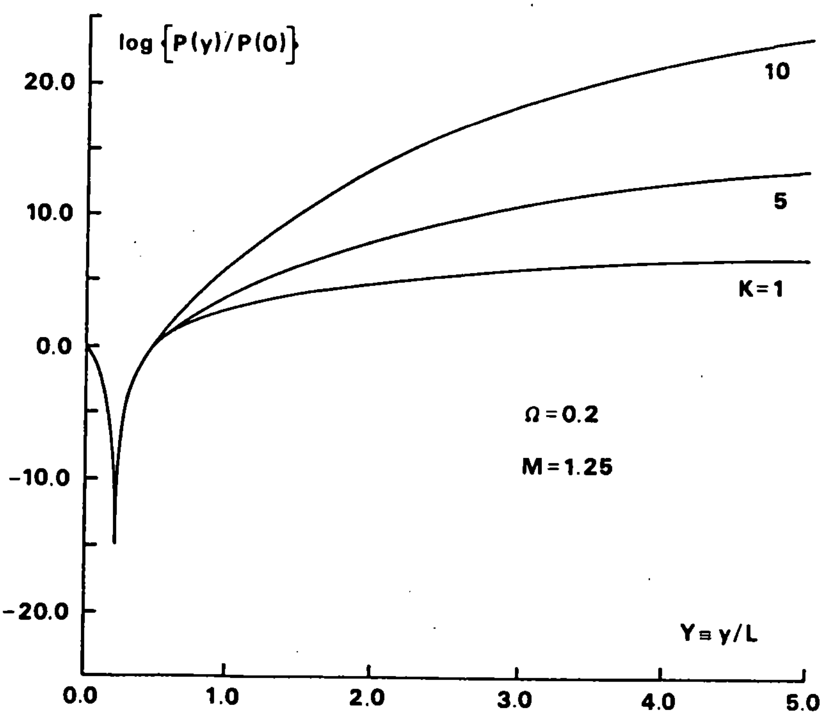

FIGURE 2