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DESCRIPTION AND ANALYSIS OF ELASTIC WAVES IN FUNCTIONALLY GRADED MATERIALS

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Functionally graded materials (FGMs) belong to a class of advanced materials characterized by variation in properties as the dimension varies. Propagation of elastic waves through FGMs is an issue of scientific and practical interest because the effective use of elastic waves in the industries relies on a good understanding of wave propagation in FGMs. The propagation of one-dimensional elastic waves in a plate made of FGMs excited by a harmonic force is described and studied in this work. The corresponding model equation is solved analytically and its solution is based on the local Heun functions. The elastic waves are investigated by means of the transmission coefficient, which can be utilized in study of transmission properties of locally periodic structures.

Keywords: Heun's equation, Heun function, functionally graded material

1. Introduction

Functionally Graded Material (FGM) belongs to a class of advanced materials with varying properties over a changing dimension. Functionally graded materials, eliminates the sharp interfaces existing in composite material which is where failure is initiated. It replaces this sharp interface with a gradient interface which produces smooth transition from one material to the next. One unique characteristics of FGM is the ability to tailor a material for specific application.

The gradient material can be conveniently described by the use of a transition function represents the volume fraction of one of the phases as a function of position. In many practical cases the compositional variation is restricted to one coordinate. Mainly, FGMs, which are compositionally graded from a ceramic phase to a metal phase, attract great attention. The ceramic/metal FGMs can be designed to reduce thermal stresses and take advantage of the heat and corrosion resistances of ceramic and the mechanical strength, high toughness good machinability and bonding capability of metals without severe internal thermal stresses. The ceramic/metal FGMs exhibit higher fracture resistance parameters resulting in higher toughness owing to crack bridging in a graded volume fraction. Due to the high mechanical and thermal properties of the constituent materials, the ceramic/metal FGMs can exhibit good service performance under some severe environments, such as super high temperature and great temperature gradient, see e.g.[1].

This work extends the current group of material-property-transient functions for which exact analytical solutions are known and it also meets the requirement of a smooth connection to the neighboring regions with constant material parameters. To accomplish this intention, we used a trigonometric transition function for the material composition in a FGM rod plate. For this function, it is possible to transform the original model equation into Heun's differential equation of which an exact solution is expressed in terms of the local Heun functions. In the last ten years we have witnessed an increased interest in Heun's equation, which is due to the fact that this equation is the most general canonized

linear second-order Fuchsian equation containing four regular singularities. This structure of singularities causes that the Heun functions as solutions of Heun's equation are increasingly appearing in the modeling of different types of problems in many areas of physics and are implemented and they are implemented in some widely used mathematical softwares, e.g. Maple.

2. Model equation

We assume a FGM rod plate of thickness d which is compositionally graded from material M_I to M_{II}. Longitudinal propagation of elastic waves through a continuously inhomogeneous rod can be described by means of the following model equation (see e.g. [2])

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{E(x)}\frac{\mathrm{d}E(x)}{\mathrm{d}x}\frac{\partial u(x,t)}{\partial x} = \frac{1}{c_L^2(x)}\frac{\partial^2 u(x,t)}{\partial t^2} \,, \tag{1}$$

where $u(x,t)$ is an axial displacement of the rod at position x and time t, and $c_L(x) = \sqrt{E(x)/\rho(x)}$ is a speed of longitudinal elastic waves.
To solve Eq. (1) it is convenient to rewrite it into its dimensionless form:

$$\frac{\partial^2 \mathcal{U}(s,\theta)}{\partial s^2} + \frac{1}{\eta(s)}\frac{\mathrm{d}\eta(s)}{\mathrm{d}s}\frac{\partial \mathcal{U}(s,\theta)}{\partial s} = \frac{1}{C^2(s)}\frac{\partial^2 \mathcal{U}(s,\theta)}{\partial \theta^2} \,, \tag{2}$$

Here

$$s = \frac{x}{\ell} \,, \quad \theta = \omega t \quad \mathcal{U} = \frac{u}{\ell} \,, \quad C(s) = \frac{c_L(s)}{\omega \ell} \,,$$

where ℓ is a characteristic length and

$$\eta(a,\sigma;s) = 1 + \frac{E_{II} - E_I}{E_I}\sin^2(s-\sigma) = 1 + a\sin^2(s-\sigma) \,, \tag{3}$$

$$\xi(b,\sigma;s) = 1 + \frac{\rho_{II} - \rho_I}{\rho_I}\sin^2(s-\sigma) = 1 + b\sin^2(s-\sigma) \,, \tag{4}$$

where E is Young's modulus, ρ is the mass density and the indexes I, II correspond to the chosen constituent materials M_I, M_{II}.
Substituting for the material-property-transient functions $\eta(s)$ and $\xi(s)$ from the relations (3), (4) and limiting ourselves to the harmonic solutions of Eq. (1), i.e. $\mathcal{U}(s,\theta) = U(s)\exp(-\mathrm{j}\theta)$, where $\mathrm{j} = \sqrt{-1}$ is the imaginary unit, we can write the model equation as

$$\frac{\mathrm{d}^2 U(s)}{\mathrm{d}s^2} + \frac{a\sin(2(s-\sigma))}{1 + a\sin^2(s-\sigma)}\frac{\mathrm{d}U(s)}{\mathrm{d}s} + K^2\frac{1 + b\sin^2(s-\sigma)}{1 + a\sin^2(s-\sigma)}U(s) = 0 \,, \tag{5}$$

where $K^2 = \omega^2\ell^2\rho_I/E_I$. The equation (5) represents a generalization of Ince's differential equation (see e.g. [3]).

3. Transformation of the model equation and its solution

To transform Eq. (5) to Heun's equation we introduce the following variable

$$z = \sin^2(s-\sigma) \,. \tag{6}$$

After some algebra, we obtain Heun's equation

$$\frac{\mathrm{d}^2 U}{\mathrm{d}z^2} + \left(\frac{1}{2z} + \frac{1}{2(z-1)} + \frac{1}{z+1/a}\right)\frac{\mathrm{d}U}{\mathrm{d}z} - \frac{bK^2 z + K^2}{4az(z-1)(z+1/a)}U(z) = 0 \,. \tag{7}$$

Based on the comparison with the canonical form of Heun's equation (see e.g. [4, 5])

$$\frac{\mathrm{d}^2U}{\mathrm{d}z^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\varepsilon}{z-q}\right)\frac{\mathrm{d}U}{\mathrm{d}z}+\frac{\alpha\beta z-g}{z(z-1)(z-q)}U(z)=0\ , \tag{8}$$

we get

$$\gamma=\frac{1}{2}\ ,\ \ \delta=\frac{1}{2}\ ,\ \ \varepsilon=1\ ,\ \ g=\frac{K^2}{4a}\ ,\ \ q=-\frac{1}{a}\ . \tag{9}$$

On the basis of the comparison with the canonical form of Heun's equation (6) and the Fuchsian condition $1+\alpha+\beta=\gamma+\delta+\varepsilon$ (see e.g [4, 5]) we obtain

$$\alpha\beta=-\frac{bK^2}{4a}\ ,\ \ \alpha+\beta=1\ . \tag{10}$$

Solving the system of equations (10) we can write

$$\alpha=\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{bK^2}{a}}\ ,\ \ \beta=\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{bK^2}{a}}\ . \tag{11}$$

The general solution of Eq. (8) is given as (see e.g [4, 5, 6])

$$\begin{aligned}U(z)=A_1H\ell(q,g;\alpha,\beta,\gamma,\delta;z)+\\ A_2z^{1-\gamma}H\ell(q,(q\delta+\varepsilon)(1-\gamma)+g;\alpha+1-\gamma,\beta+1-\gamma,2-\gamma,\delta;z)\ ,\end{aligned} \tag{12}$$

where A_1, A_2 are integration constants and $H\ell$ represents the local Heun function with appropriate parameters (see e.g [4, 5, 6]).
Substituting from the relations (6), (9) and (11) into the solution (12) we obtain the general solution of Eq. (5)

$$\begin{aligned}U(s)=A_1H\ell\left(-\frac{1}{a},\frac{K^2}{4a};\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{bK^2}{a}},\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{bK^2}{a}},\frac{1}{2},\frac{1}{2};\sin^2(s-\sigma)\right)+\\ A_2\sin(s-\sigma)\,H\ell\left(-\frac{1}{a},\frac{2a-1+K^2}{4a};1+\frac{1}{2}\sqrt{1+\frac{bK^2}{a}},1-\frac{1}{2}\sqrt{1+\frac{bK^2}{a}},\right.\\ \left.\frac{3}{2},\frac{1}{2};\sin^2(s-\sigma)\right)\equiv A_1H_{11}(s)+A_2\sin(s-\sigma)\,H_{12}(s)\ ;\ \sin^2(s-\sigma)<1\ .\end{aligned} \tag{13}$$

As the local Heun functions are evaluable only for their arguments from the interval $[0,1)$ the condition $\sin^2(s-\sigma)<1$ in Eq. (13) has to be satisfied, see e.g. [4].

Without loss of generality, we will search for the solution of Eq. (13) in the closed interval $[0,\pi/2]$, which enables us to fulfill the requirement concerning the smooth connection of the inhomogeneous region to the neighboring homogeneous regions. However, it is necessary to pay attention to the fact that the solution (13) is valid only in the half-closed interval $[0,\pi/2)$ where the absolute value of the argument of the Heun functions is less than 1, which is the regular singular point.

To overcome the problem we can employ the ensuing variable transformation

$$y=1-z\ , \tag{14}$$

which means that the neighborhood of $z=1$ is transformed into the neighborhood of $y=0$. Using the transformation (14) Heun's equation (8) can be rewritten for the parameters $(q,g,\alpha,\beta,\gamma,\delta)$ into the form

$$\frac{\mathrm{d}^2U}{\mathrm{d}y^2}+\left(\frac{\tilde{\gamma}}{y}+\frac{\tilde{\delta}}{y-1}+\frac{\tilde{\varepsilon}}{y-\tilde{q}}\right)\frac{\mathrm{d}U}{\mathrm{d}y}+\frac{\tilde{\alpha}\tilde{\beta}y-\tilde{g}}{y(y-1)(y-\tilde{q})}U(y)=0\ , \tag{15}$$

where

$$\tilde{q} = 1 - q \ , \quad \tilde{g} = \alpha\beta - g \ , \quad \tilde{\alpha} = \alpha \ , \quad \tilde{\beta} = \beta \ , \quad \tilde{\delta} = \gamma \ , \quad \tilde{\gamma} = \delta \ . \tag{16}$$

The general solution of Eq. (15) is

$$\begin{aligned} U(s) = B_1 H\ell\left(\tilde{q}, \tilde{g}; \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}; \cos^2(\pi s - \sigma)\right) + \\ B_2 \cos(\pi s - \sigma) H\ell\left(\tilde{q}, (\tilde{q}\tilde{\delta} + \tilde{\varepsilon})(1 - \tilde{\gamma}) + \tilde{g}; \tilde{\alpha} + 1 - \tilde{\gamma}, \tilde{\beta} + 1 - \tilde{\gamma}, 2 - \tilde{\gamma}, \tilde{\delta}; \cos^2(\pi s - \sigma)\right) = \\ B_1 H_{21}(s) + B_2 \cos(\pi s - \sigma) H_{22}(s) \ , \quad |\cos^2(\pi s - \sigma)| < 1 \ , \end{aligned} \tag{17}$$

where B_1, B_2 are integration constants. This solution represents the solution of the model equation (13) around the point $s = \pi/2$ including this point. To employ both the solutions we will divide the closed interval $[0, \pi/2]$ into two intervals $[0, s_1]$ and $[s_1, \pi/2]$ where $0 < s_1 < 1/2$. For the first interval we have the solution (13) and for the second one, we have the solution (17).

3.1 Illustrative example

As an illustrative example that demonstrates the applicability of the above-mentioned results we can solve Eq. (13) on the closed interval $[0, \pi/2]$ for the parameter $\sigma = 0$ and initial value conditions:

$$U(0) = 1 \quad \text{and} \quad \left.\frac{\mathrm{d}U(s)}{\mathrm{d}s}\right|_{s=0} = 0 \ . \tag{18}$$

Using the boundary conditions we obtain $A_1 = 1$ and $A_2 = 0$. Then, we can calculate the integration

	$M_I = Al$ Aluminium	$M_{II} = Al_2O_3$ Alumina
E (GPa)	70.0	393.0
ρ (kgm^{-3})	2707	3960

Table 1: Material properties of considered constituents.

constants B_1 and B_2 based on the continuity conditions of the solutions (13) and (17) at the point $0 < s_1 < \pi/2$, i.e.

$$A_1 H_{11}(s_1) = B_1 H_{21}(s_1) + B_2 \cos(\pi s_1) H_{22}(s_1) \tag{19}$$

and

$$A_1 \left.\frac{\mathrm{d}H_{11}(s)}{\mathrm{d}s}\right|_{s=s_1} = \left.\frac{\mathrm{d}}{\mathrm{d}s}(B_1 H_{21}(s) + B_2 \cos(\pi s) H_{22}(s))\right|_{s=s_1} \ . \tag{20}$$

The Heun functions are evaluated for the material parameters from Tab. 1. Solving Eqs. (19) and (20) we obtain values of the integration constants B_1 and B_2. The solution of Eq. (13) and its derivative for the whole interval $[0, \pi/2]$ is depicted in Fig. 1. We used the mathematical software Maple in which the local Heun functions and their derivatives are implemented.

4. Transfer matrix calculation

Employing the same approach as in the previous section it is possible to calculate the transfer matrix, however it is again necessary to take into account the fact that the point $\pi/2$ is the regular singular one. In the following text, we will consider the parameter σ to be equal to 0.
Based on the solution (13) we can write

$$\begin{pmatrix} U(s) \\ U'(s) \end{pmatrix} = \mathbf{T}^{(1)}(s, s_0) \begin{pmatrix} U(s_0) \\ U'(s_0) \end{pmatrix} \ ; \quad s_0 = 0 \leq s < \frac{\pi}{2} \ , \tag{21}$$

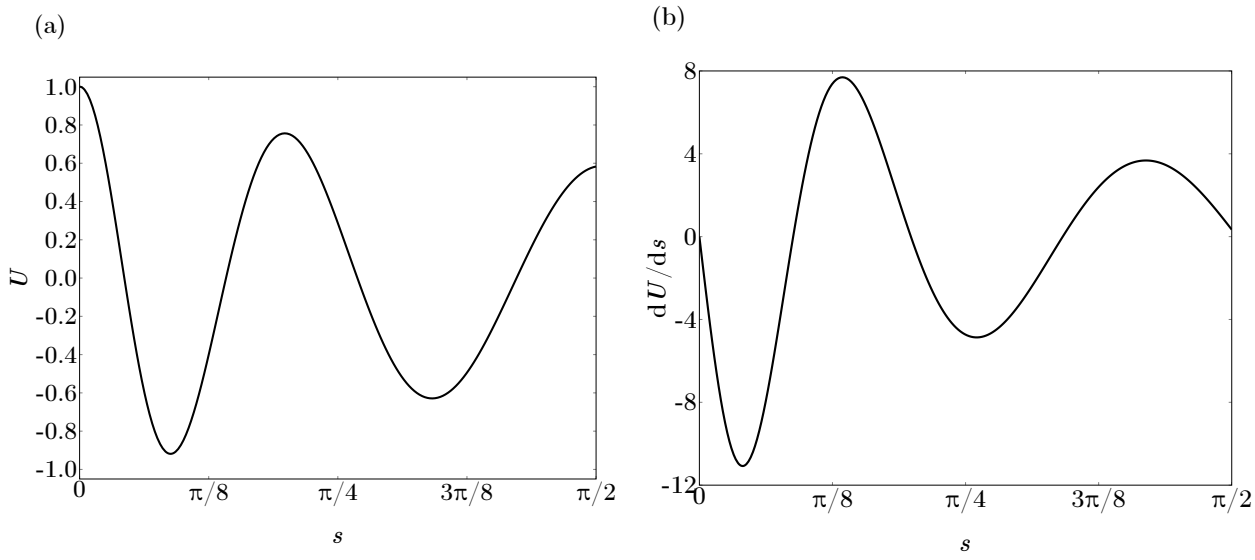

Figure 1: Solution of Eq. (5) for the parameter $\sigma = 0$, the material constituents from Tab. 1, and the initial value conditions (18) (a) and its derivative (b).

where the prime denotes the derivative with respect to s and $\mathbf{T}^{(1)}$ is the transfer matrix

$$\mathbf{T}^{(1)}(s, s_0) = \begin{pmatrix} \frac{H_{11}(s)H'_{12}(s_0) - H_{12}(s)H'_{11}(s_0)}{H_{11}(s_0)H'_{12}(s_0) - H_{12}(s_0)H'_{11}(s_0)} & \frac{-H_{11}(s)H_{12}(s_0) + H_{12}(s)H'_{11}(s_0)}{H_{11}(s_0)H'_{12}(s_0) - H_{12}(s_0)H'_{11}(s_0)} \\ \frac{H'_{11}(s)H'_{12}(s_0) - H'_{12}(s)H'_{11}(s_0)}{H_{11}(s_0)H'_{12}(s_0) - H_{12}(s_0)H'_{11}(s_0)} & \frac{-H'_{11}(s)H_{12}(s_0) + H'_{12}(s)H_{11}(s_0)}{H_{11}(s_0)H'_{12}(s_0) - H_{12}(s_0)H'_{11}(s_0)} \end{pmatrix} .$$

Supposing a point s_1 from the interval $0 < s_1 < \pi/2$ then we have

$$\begin{pmatrix} U(s) \\ U'(s) \end{pmatrix} = \mathbf{T}^{(2)}(s, s_1) \begin{pmatrix} U(s_1) \\ U'(s_1) \end{pmatrix} , \quad s_1 \leq s \leq \pi/2 , \tag{22}$$

where

$$\mathbf{T}^{(2)}(s, s_1) = \begin{pmatrix} \frac{H_{21}(s)H'_{22}(s_1) - H_{22}(s)H'_{21}(s_1)}{H_{21}(s_1)H'_{22}(s_1) - H_{22}(s_1)H'_{21}(s_1)} & \frac{-H_{21}(s)H_{22}(s_1) + H_{22}(s)H_{21}(s_1)}{H_{21}(s_1)H'_{22}(s_1) - H_{22}(s_1)H'_{21}(s_1)} \\ \frac{H'_{21}(s)H'_{22}(s_1) - H'_{22}(s)H'_{21}(s_1)}{H_{21}(s_1)H'_{22}(s_1) - H_{22}(s_1)H'_{21}(s_1)} & \frac{-H'_{21}(s)H_{22}(s_1) + H'_{22}(s)H_{21}(s_1)}{H_{21}(s_1)H'_{22}(s_1) - H_{22}(s_1)H'_{21}(s_1)} \end{pmatrix} . \tag{23}$$

Then, the transfer matrix for the whole interval $[0, \pi/2]$ can be calculated based on the two transfer matrices (22) and (23)

$$\mathbf{T} = \mathbf{T}^{(2)}(\pi/2, s_1)\mathbf{T}^{(1)}(s_1, 0) . \tag{24}$$

By means of the overall transfer matrix (24) we can calculate the reflection and transmission coefficients for wave propagation through the region $[0, \pi/2]$.
The above mentioned calculations of the transfer matrices are also easy applicable for the case of $\sigma = \pi/2$.

5. Conclusion

In this work, we dealt with longitudinal elastic waves propagating through a rod plate that is fabricated of FGM. We assumed the material composition in the FGM plate to be distributed according to a trigonometric transient function that ensures the zero derivatives of material density and Young's modulus at the endpoints, in which the plate is connected to the homogeneous regions. We analytically solved the equation which models elastic waves in the FGM plate. The model equation represents the generalized form of Ince's differential equation that can be transformed to the canonical

form of Heun's equation. The general solution of this equation is expressed by means of the local Heun functions. To accomplish the requirement concerning smooth connections to neighboring homogeneous regions it is necessary to evaluate the solution of Heun's equation on the closed interval which includes two regular singular points. We presented a method that enables us to resolve this problem. To demonstrate the applicability of the method we presented illustrative example of the model equation solution for given material constituents. In addition, based on this method we also present transfer matrix calculation.

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