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"Error bounds fo igenvalue analysis by elimination of variables".

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1. Introduction

The dynamic analysis of a structure by the finite element method leads often to an eigenvalue problem of such a magnitude that its eigensolutions are very expensive to obtain. Indeed the structural dynamic flexibility matrix being fully populated it is not possible to use the same partitioning methods as in statics(1).

Two approaches have been used to solve these large eigenvalue problems. The first one consists in seeking the stationarity points of the Rayleigh quotient by minimization techniques (2,3). The second aims at an approximation preserving the low frequency spectrum while reducing the order of the eigenvalue problem. Several authors have followed such an approach (4,5); the most powerful one has been proposed by R.G. ANDERSON, B.M. IRONS and O.C. ZIENKIEWICZ. It consists in eliminating those displacements that give no appreciable contribution to the kinetic energy. When using the finite element method, this algorithm allows an easy step by step build up of the whole structure (6) : each elimination operation produces free place for the next assembling sequence.

The numerical results obtained show that, when the choice of remaining displacements is guided by some engineering skill, the loss of accuracy can be very small.

Furthermore, it can be proved, as a consequence of the "Maximum-minimum property of eigenvalues" established by Courant (7), that the eigenvalues are always increased by the reduction process.

It seems however important to evaluate the loss of accuracy caused by the elimination process on the basis of a numerical criterion. Such a bound algorithm has been proposed by C.C. WRIGHT and G.A. MILES (8). As will be shown, more accurate error bounds for eigenvalue determination can be obtained from an application of the theorems of T. KATO and G. TEMPLE (9,10).

These ideas will be applied to a large scale problem : a delta wing analyzed experimentally and numerically by R.J.J. TABOREK (11), the idealization of which involves a large number of degrees of freedom.

2. Elimination of variables (5,6,8)

The matrix equation giving the natural circular frequencies ω and the modal shapes q of the structure is

$$Kq = \omega^2 Mq \qquad (2.1)$$

where the structural stiffness and mass matrices K and M result from a convenient addressing of the elementary matrices k_e and m_e.

The technique used to solve (2.1) consists in choosing some displacements q_c to be eliminated, regardless of the corresponding alteration of the kinetic energy. If we denote by q_R the remaining displacements, (2.1) can be partitionned as follows:

$$\begin{bmatrix} K_{RR} & K_{RC} \\ K_{CR} & K_{CC} \end{bmatrix} \begin{bmatrix} q_R \\ q_C \end{bmatrix} = \omega^2 \begin{bmatrix} M_{RR} & M_{RC} \\ M_{CR} & M_{CC} \end{bmatrix} \begin{bmatrix} q_R \\ q_C \end{bmatrix} \qquad (2.2)$$

By assuming that the q_C give no contribution to the kinetic energy, (2.2) reduces to

$$\bar{K}_{RR}\, q_R = \bar{\omega}^2\, \bar{M}_{RR}\, q_R \qquad (2.3)$$

with

$$\bar{K}_{RR} = K_{RR} - K_{RC}\, K_{CC}^{-1}\, K_{CR} \qquad (2.4)$$

and

$$\bar{M}_{RR} = M_{RR} + K_{RC}\, K_{CC}^{-1}\, M_{CC}\, K_{CC}^{-1}\, K_{CR} - K_{RC}\, K_{CC}^{-1}\, M_{CR} - M_{RC}\, K_{CC}^{-1}\, K_{CR} \qquad (2.5)$$

The eliminated displacements are restitued by the relation

$$q_C = - K_{CC}^{-1}\, K_{CR}\, q_R \qquad (2.6)$$

It will be shown how this algorithm can be used when the structural eigenvalue problem (2.3) is set up by coupling of substructures.

3. Alteration of the eigensolution

The error analysis of the reduction algorithm shows that the loss of accuracy is governed by the ratio $\left(\frac{\omega}{\mu_1}\right)^2$, where μ_1^2 denotes the first eigenvalue of the interior eigenvalue problem

$$K_{CC}\, q_C = \mu^2\, M_{CC}\, q_C \qquad (3.1)$$

Therefore the displacements q_C to be condensed should be selected in the structural regions of least dynamic flexibility.

It will also be proved, as a consequence of the general maximum-minimum property of eigenvalues established by Courant (7), that the eigensolution of (2.3) furnishes an upper bound to that of (2.1).

4. Bound algorithm

The loss of accuracy in the reduction process can be measured by computing upper and lower bounds to exact eigenvalues of (2.1).

Let W_0 be an approximation to an eigenmode $q_{(i)}$ associated to the eigenvalue ω_i^2 of (2.1). The corresponding Rayleigh quotient can be written as

$$\rho = \frac{W_0'\, K\, W_0}{W_0'\, M\, W_0} \qquad (4.1)$$

A first bound algorithm, due to Krylov and Bogoliubov (13), produces numbers λ_i^- and λ_i^+ such that

$$\lambda_i^- < \omega_i^2 < \lambda_i^+ \qquad (4.2)$$

if we know the Rayleigh quotient (4.1) and the first iterate of W_0 defined as

$$W_1 = K^{-1}\, M\, W_0 \quad . \qquad (4.3)$$

More accurate bounds can be obtained as an application of theorems of T. KATO and G. TEMPLE (14,15). It is therefore necessary to determinate two numbers μ and ν such that

$$\omega_{i-1}^2 \leq \mu < \omega_i^2 < \nu \leq \omega_{i+1}^2 \qquad (4.4).$$

The approximate bounds μ and ν to the adjacent eigenvalues ω_{i-1}^2 and ω_{i+1}^2 can be obtained by use of the first bound algorithm.

5. The substructure technique

The reduction process described in section 2 allows a step by step assembling process for the whole structure. Indeed each elimination operation produces free space in core storage of the computer for the next assembling sequence.

After the resolution of the reduced eigenvalue problem (2.3), the approximate modes obtained can be restitued into the whole set of structural displacements : for each substructure, the condensed displacements are computed by (2.6).

It will also be shown that the first iterate (4.3) of each eigenmode can be computed without assembling again the whole structure. Indeed the elements needed to solve the static problem

$$K\, W_1 = \bar{\omega}^2\, M\, W_0 \qquad (5.1)$$

can be memorized on peripheric devices when eliminating displacements.

6. Numerical application and conclusions

In order to prove its computational efficiency, the reduction method has been applied to a large scale problem : a delta wing analized experimentally and numerically by R.J.J. TABOREK (11) and represented by the fig. 1.

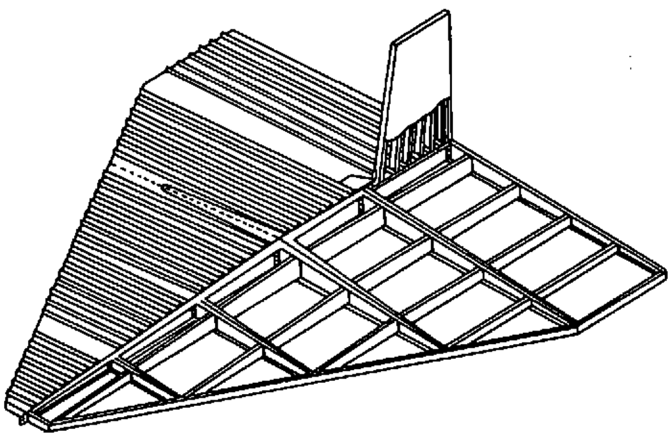

Fig. 1. Idealization of the Delta WING

The whole structure is treated as a Kirchhoff plate : a specially adapted conforming plate element has been generated which takes into account the variable distance between the upper and lower skins, and the anisotropy introduced by the skin stiffners.

The loss of accuracy produced by the reduction method is evaluated by use of the bound algorithm of T. KATO and G. TEMPLE. At an other side, the comparison with the complete eigensolution furnished by the minimization techniques (3) give another verification.

The results obtained prove that the reduction method can be considered as the most economical way to solve large eigenvalue problems.

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