24th INTERNATIONAL CONGRESS ON SOUND AND VIBRATION
23–27 July 2017, London

FREQUENCY ANALYSIS OF NONLINEAR STRUCTURES WITH UNCERTAINTIES PARAMETERS USING POLYNOMIAL CHAOS EXPANSION

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In this study, we propose a formulation of the problem of non-linear dynamics by considering the random geometric or mechanical physical parameters and by developing the solution on the basis of polynomial chaos. This approach coupled Harmonic Balance Method (HBM) with the Polynomial Chaos Expansion (PCE) method. This approach developed will be called stochastic HBM. The difficulty associated with the evaluation of nonlinearity in both the frequency domain and the stochastic domain will be resolved. Several numerical simulations illustrate the efficiency and the precision of the proposed method.

Keywords: nonlinear, uncertainties, PCE, HBM.

1. Introduction

The study of non-linear systems takes a prominent place in many industrial fields such as auto-motive and aerospace. This importance has motivated many scientists to predict the dynamic behaviour of nonlinear systems. In addition, it has been shown that geometric and material physical parameters admit dispersions linked in general to the manufacturing processes. Therefore it is necessary to take these dispersions account in order to predict a robust dynamic behaviour capable of assisting a robust design of these systems.

Many methods are taken into account in the prediction and analysis of the dynamic behaviour of nonlinear systems. The Monte Carlo Simulation (MCS) method is the reference method [1]. This method is based on the resolution of simulations for different values of the random parameters, requires many realizations and therefore proves costly in computation time. In this context, the Polynomial Chaos Expansion (PCE) method can be a very interesting alternative [2]. It is based on a probabilistic characterization of the uncertainties; it formalizes a stochastic process with finite variance by a series development of polynomial functions. The last are orthogonal to a defined probability measure with respect to some independent random variables modelling the uncertainties. The Polynomial Chaos Expansion (PCE) method has already proved its effectiveness in linear problems [3]. Nevertheless, the latter presents difficulties with non-linear problems presenting multiple solutions.

The objective of this study is to estimate the stochastic nonlinear dynamic response and this for a reasonable computation cost. A method based on the coupling (PCE) of and Harmonic Balance Method (HBM) will be developed. Indeed, (HBM) has proved its efficiency in dealing with non-linear problems [4], while the (PCE) method allows dealing random problems. First, the modelling of the system with the non-linearity will be detailed. After, random phenomena will be explained. Finally, the solution of the posed problem will be detailed. We will recall the methodology for dealing with the deterministic non-linear case, and then we will look at systems with regular uncertainties and nonlinearities. The proposed method will be validated using numerical simulations in several case studies.

2. Modeling of a non–linear structure

The discretization by the finite element method with N Degree Of Freedom (DOF) of a linear structure gives the following matrix system:

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F_{ext}\} \tag{1}$$

With $[M]$ mass matrix $[C]$ damping matrix $[K]$ stiffness matrix and we note $\{u\}$ the displacement vector and $\{F_{ext}\}$ the external forces vector.
In this system there may be non-linear phenomena due to:

- Intermittent contact or friction called non-linearity of contact modeled by non-linear force:

$$\{F_{nl}(\{u\})\} = \begin{cases} k_1\{u\} & \text{if}\{u\} > 0 \\ k_2\{u\} & \text{if}\{u\} < 0 \end{cases} \tag{2}$$

- Large displacement for thin structures called geometric nonlinearity modeled by non-linear force:

$$\{F_{nl}(\{u\})\} = [K_{nl}]\{u^p\} \tag{3}$$

  p essentially quadratic or cubic.

The nonlinear system modeling the nonlinear dynamics of a structure discretized by the finite element method is given by:

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} + \{F_{nl}(\{u\})\} = \{F_{ext}\} \tag{4}$$

with $\{F_{nl}(\{u\})\}$ is the non linear force vector. In this study, we used a cubic geometric non linearity.

3. Stochastic non–linear problem solving

3.1 Deterministic non–linear formulation

In order to solve the non-linear problem defined in Equation (4), we propose to use Harmonic Balance method (HBM) method. This approach consists in expressing the non-linear dynamic response $\{u(t)\}$ as well as the excitation $\{F_{ext}\}$ in the form of multiple Fourier series [5]:

$$\Big\{u(t)\Big\} = \Big\{\sum_{p=1}^{H}\Big(U_p^c\cos(p\omega t) + U_p^s\sin(p\omega t)\Big)\Big\} \tag{5}$$

$$\Big\{F_{ext}\Big\} = \Big\{\sum_{p=1}^{H}\Big(F_p^c\cos(p\omega t) + F_p^s\sin(p\omega t)\Big)\Big\} \tag{6}$$

with H the number of harmonics retained. The hypothesis on H is based on the a priori knowledge of the nature of $\{u(t)\}$. Most often, the mechanical systems react to the 3rd, see the 5th harmonic and U_p^c and U_p^s define the unknown coefficients of the finite Fourier series. The (HBM) allows only periodic or quasi–periodic responses to be processed. For chaotic responses temporal integration remains the only solution. To illustrate the method, we take the case of geometric non–linearity. Non-linearity results in a displacement-dependent stiffness matrix. Equation (4) becomes:

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K_{nl}]\{u\} = \{F_{ext}\} \tag{7}$$

with

$$[K_{nl}]\{u\} = [K]\{u\} + \{F_{nl}(\{u\})\} \tag{8}$$

By injecting the equation (5) and (6) into equation (7), and by isolating the contributions of each harmonic, we obtain a nonlinear algebraic system of size $2 \times H \times N$:

$$\begin{pmatrix} \ddots & [0] & \cdots \\ [0] & \begin{bmatrix} -(p\omega)^2[M] + K_{nl}(\{u\})] & p\omega[C] \\ -p\omega[C] & -(p\omega)^2[M] + K_{nl}(\{u\})] \end{bmatrix} & [0] \\ \cdots & [0] & \ddots \end{pmatrix} \begin{Bmatrix} \vdots \\ U_p^c \\ U_p^s \\ \vdots \end{Bmatrix} = \begin{Bmatrix} F_p^c \\ F_p^s \end{Bmatrix} \tag{9}$$

The perturbation method often used to solve the Duffing equation is not suited to large problems. The aim being to follow the evolution of the vibratory behaviour as a function of the pulsion ω, the problem Eq (4) is solved in the form:

$$R(U,\omega) = L(U,\omega)U - F = 0 \tag{10}$$

where L is defined by $L = diag(L_1, \dots, L_p, \dots\dots\dots L_H)$,
with

$$L_p = \begin{bmatrix} -(p\omega)^2[M] + K_{nl}(\{u\})] & p\omega[C] \\ -p\omega[C] & -(p\omega)^2[M] + K_{nl}(\{u\})] \end{bmatrix} \tag{11}$$

Then U and, F denote the unknown vector of harmonic coefficients, the projection of the external forces. They are given by respectively:

$$U = [U_1^c, U_1^s, \dots\dots, U_p^c, U_p^s, \dots\dots, U_H^c, U_H^s] \tag{12}$$

$$F = [F_1^c, F_1^s, \dots\dots, F_p^c, F_p^s, \dots\dots, F_H^c, F_H^s] \tag{13}$$

R is a non–linear regular function which depends on the parameters ω and the unknown U. The non–linear behavior generates the possibility of having several solutions of U for one ω given. The continuation method has been developed to solve this type of equation. The resolution is iteratively following a certain path that makes the solution unique to iteration, Continuation methods are classified into two categories: the Prediction–Correction (PC) continuation method [6] and the Numerical Asymptotic Method (NAM) [7]. In this work, I have opted for the Newton–Raphson numerical algorithm for solving the non linear algebraic Equation (10), the

curve (U, ω) can be obtained as sequence of points for discrete value of ω.
Appling the Galerkin method, for simplify we will trounced the harmonic series to fundamental harmonic $p = 1$, the residual is orthogonalized with respect the trial functions of harmonic approximation $\sin \omega t$ and $\cos \omega t$.

$$\int_0^{\frac{2\pi}{\omega}} R_i(t) \sin(\omega t) dt = 0 \qquad \text{for} \qquad i = 1 \ldots\ldots H \times N \tag{14}$$

$$\int_0^{\frac{2\pi}{\omega}} R_i(t) \cos(\omega t) dt = 0 \qquad \text{for} \qquad i = 1 \ldots\ldots H \times N \tag{15}$$

Then we obtained:

$$G(U_i^c, U_i^s, \omega) = 0 \tag{16}$$

The following Newton–Raphson approximation is used:

$$(U_i^{cj+1}, U_i^{sj+1}) = (U_i^{cj}, U_i^{sj}) + G^{-1}_{(U_i^c, U_i^s)}(U_i^{cj}, U_i^{sj}) G_{(U_i^c, U_i^s)}(U_i^{cj}, U_i^{sj}) \tag{17}$$

where $G_{(U_i^c, U_i^s)}$ is the jacobian of $G(U_i^c, U_i^s)$.
The general principle of these methods is to find a solution $(U_i^{cj+1}, U_i^{sj+1}, \omega^{j+1})$ satisfying the criterion $\|R(U^{j+1}, \omega^{j+1})\| < \varepsilon$, (U^{j+1}, ω^{j+1}) is generated from the solution (U^j, ω^j) through a prediction followed by a succession of corrections.

3.2 Stochastic response of the system based on the Polynomial Chaos Expansion

In a general context several forces and materials parameters can be considered as random, the following random quantities have to be considered: mass, damping and stiffness matrix parameters (defined respectively by $[\widetilde{M}]$, $[\widetilde{C}]$ and$[\widetilde{K}]$) as well as the external forces terms defined by $\{\widetilde{F}ext\}$. For the non-linear system, the equation of motion Eq (4) may be written as:

$$[\widetilde{M}]\{\ddot{\widetilde{u}}\} + [\widetilde{M}]\{\dot{\widetilde{u}}\} + [K]\{\widetilde{u}\} + \{F_{nl}(\{\widetilde{u}\})\} = \{\widetilde{F}_{ext}\} \tag{18}$$

We propose to expand the uncertain parameters by using the classical Karhunen–Loeve expansion [8] with the Galerkin formulation of the finite element method [9] as follows:

$$[\widetilde{K}] = \sum_{i=0}^{K} [K_i] \cdot \xi_i, \qquad [\widetilde{C}] = \sum_{j=0}^{C} [C_j] \cdot \xi_j, \qquad [\widetilde{M}] = \sum_{k=0}^{M} [M_k] \cdot \xi_k \tag{19}$$

The external vector force is:

$$\{\widetilde{F}_e\} = \sum_{m=0}^{F} \{f_{em}\} \cdot \xi_m$$

ξ_i,ξ_j, ξ_m and ξ_k are the random variables.
The response of non linear dynamic systems with the random properties is also a random process the vectors $u(t)$, $\dot{u}(t)$ and $\ddot{u}(t)$ are expanded along polynomial chaos basis.

$$\{\widetilde{u}(t)\} = \sum_{n=0}^{U} \{u_n(t)\} \cdot \psi_n(\{\xi_i\}_{i=1}^{Q})$$
$$\{\dot{\widetilde{u}}(t)\} = \sum_{n=0}^{U} \{\dot{u}_n(t)\} \cdot \psi_n(\{\xi_i\}_{i=1}^{Q})$$
$$\{\ddot{\widetilde{u}}(t)\} = \sum_{n=0}^{U} \{\ddot{u}_n(t)\} \cdot \psi_n(\{\xi_i\}_{i=1}^{Q}) \tag{20}$$

where $\psi(\xi_i)$ are multidimensional Hermit orthogonal polynomials of the random variables ξ_i defined by:

$$\psi_n(\xi_i,\dots\dots\dots\xi_p) = (-1)^p \cdot \exp\left(\frac{1}{2}\,^T\{\xi\}\{\xi\}\right) \frac{\partial^p\Big(-\frac{1}{2}^T\{\xi\}\{\xi\}\Big)}{\partial\xi_i \dots\dots\dots \partial\xi_p} \tag{21}$$

$u_n(t)$, $\dot{u}_n(t)$ and $\ddot{u}_n(t)$ denote a vector determinist coefficients Subsisting all this development into equation of motion:

$$\sum_{k=0}^{M}\sum_{n=0}^{U}\{\ddot{u}_n(t)\}\cdot[M_k]\cdot\xi_k\psi_n + \sum_{j=0}^{C}\sum_{n=0}^{U}\{\dot{u}_n(t)\}\cdot[C_j]\cdot\xi_j\psi_n$$
$$+\sum_{i=0}^{K}\sum_{n=0}^{U}\{u_n(t)\}\cdot[K_i]\cdot\xi_i\psi_n + f_{nl}\Big(\sum_{n=0}^{U}\{u_n(t)\}\cdot\psi_n\Big) = \sum_{m=0}^{F}[f_e]\cdot\xi_m \tag{22}$$

we multiply the equation obtained by ψ_m. If we used averaged (integration on the domain of random variables), and use the orthogonality properties of polynomials, we obtained the following equation:

$$\sum_{k=0}^{M}\sum_{n=0}^{U}\{\ddot{u}_n(t)\}\cdot[M_k]\cdot <\xi_k\,\psi_n\,\psi_m> + \sum_{j=0}^{C}\sum_{n=0}^{U}\{\dot{u}_n(t)\}\cdot[C_j]\cdot <\xi_j\,\psi_n\,\psi_m>$$
$$+\sum_{i=0}^{K}\sum_{n=0}^{U}\{u_n(t)\}\cdot[K_i]\cdot <\xi_i\,\psi_n\,\psi_m> + f_{nl}\Big(\sum_{n=0}^{U}\{u_n(t)\}\Big)\cdot <\psi_n\,\psi_m> = \sum_{m=0}^{F}[f_e]\cdot <\xi_m\,\psi_m> \tag{23}$$

$<\xi_i\,\psi_n\,\psi_m>$ is the inner product defined by the mathematical expectation operator.
Using matrix notations the resulting algebraic non linear system can be rewritten as:

$$[MG]\{\ddot{U}\} + [CG]\{\dot{U}\} + [KG]\{U\} + \{FG_{nl}\} = \{FG_e\} \tag{24}$$

with

$$\{\ddot{U}_G\} = \begin{Bmatrix} \ddot{u}_0 \\ \ddot{u}_1 \\ \vdots \\ \ddot{u}_t \\ \vdots \\ \ddot{u}_U \end{Bmatrix} \quad \{\dot{U}_G\} = \begin{Bmatrix} \dot{u}_0 \\ \dot{u}_1 \\ \vdots \\ \dot{u}_t \\ \vdots \\ \dot{u}_U \end{Bmatrix} \quad \{U_G\} = \begin{Bmatrix} u_0 \\ u_1 \\ \vdots \\ u_t \\ \vdots \\ u_U \end{Bmatrix} \quad \{FG_{nl}\} = \begin{Bmatrix} FG_{nl\,0} \\ FG_{nl\,1} \\ \vdots \\ FG_{nl\,t} \\ \vdots \\ FG_{nl\,U} \end{Bmatrix}$$

and

$$\{FG_e\} = \begin{Bmatrix} FG_{e\,0} \\ FG_{e\,1} \\ \vdots \\ FG_{e\,t} \\ \vdots \\ FG_{nl\,U} \end{Bmatrix}$$

$$[MG]^{st} = \sum_{k=0}^{M}[M_k]\cdot <\xi_k\,\psi_s\,\psi_t>\,,\ [CG]^{st} = \sum_{j=0}^{C}[C_j]\cdot <\xi_j\,\psi_s\,\psi_t>\,,\ [KG]^{st} = \sum_{i=0}^{K}[K_i]\cdot <\xi_i\,\psi_s\,\psi_t>$$

$$\{FG_{nl\,t}\} = f_{nl}\left(\sum_{n=0}^{U}\{u_n(t)\}\right)\cdot <\psi_n\,\psi_t>\ ,\ \{FG_{e\,t}\} = \sum_{m=0}^{F}\{F_{ext\,m}\}\cdot <\psi_m\,\psi_t>$$

Note that due to the orthogonality of polynomials, most of expressions $< \xi_i \, \psi_n \, \psi_m >$ are zero. We use Harmonic Balance method (HBM) method. This approach consists in expressing the non-linear stochastic dynamic response $\{U_G(t)$ as well as the excitation $\{F_{Gext}(t)\}$ in the form:

$$\{U_G(t)\} = \left\{ \sum_{p=1}^{H} (U_{Gp}^c \cos(p\omega t) + U_{Gp}^s \sin(p\omega t)) \right\} \tag{25}$$

$$\{F_{G\,ext}(t)\} = \left\{ \sum_{p=1}^{H} (F_{Gp}^c \cos(p\omega t) + F_{Gp}^s \sin(p\omega t)) \right\} \tag{26}$$

Non–linearity results in a displacement-dependent stiffness matrix. Equation (24) becomes:

$$[MG]\{\ddot{U}\} + [CG]\{\dot{U}\} + [KG_{nl}]\{U\} = \{FG_{ext}\} \tag{27}$$

with $[KG_{nl}(\{U_G\})]\{U_G\} = [KG]\{U_G\} + \{FG_{nl}(\{U_G\})\}$. By injecting the equation (25) and (26) into equation (27), and by isolating the contributions of each harmonic, we obtain a nonlinear algebraic system:

$$\begin{pmatrix} \ddots & & [0] & & \cdots \\ [0] & \begin{bmatrix} -(p\omega)^2[MG] + KG_{nl}(\{U_G\}) & p\omega[CG] \\ -p\omega[CG] & -(p\omega)^2[MG] + KG_{nl}(\{U_G\}) \end{bmatrix} & & & [0] \\ \cdots & & [0] & & \ddots \end{pmatrix} \begin{Bmatrix} \vdots \\ U_{Gp}^c \\ U_{Gp}^s \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \vdots \\ FG_p^c \\ FG_p^s \\ \vdots \end{Bmatrix} \tag{28}$$

$$R(U_G, \omega) = L(X, \omega)X - A = 0 \tag{29}$$

Then X denote the unknown vector of harmonic coefficients, and A the projection of the external forces. They are given respectively by:

$$X = [U_{G1}^c, U_{G1}^s, \ldots\ldots, U_{Gp}^c, U_{Gp}^s, \ldots\ldots, U_{GH}^c, U_{GH}^s]$$
$$A = [FG_1^c, FG_1^s, \ldots\ldots, FG_p^c, FG_p^s, \ldots\ldots, FG_H^c, FG_H^s]$$

L is defined by $L = \text{diag}(L_1, \ldots\ldots, L_p, \ldots\ldots, L_H)$.
with

$$L_p = \begin{bmatrix} -(p\omega)^2[M] + K_{nl}(\{u\}) & p\omega[C] \\ -p\omega[C] & -(p\omega)^2[M] + K_{nl}(\{u\}) \end{bmatrix}$$

After resolving equation, the mean and variance values of amplitude for p-th DOF are given directly by:

$$\text{mean}(A_p) = \sqrt{(u_{0p}^c)^2 + (u_{0p}^s)^2} \tag{30}$$

$$\text{var}(A_p) = \sum_{i=1}^{U} \left(\sqrt{(u_{ip}^c)^2 + (u_{ip}^s)^2} \right)^2 \cdot < \psi_i^2 > \tag{31}$$

3.3 Numerical examples

3.3.1 Description of the nonlinear model under study

In this section, in order to verify the applicability of the proposed method, we will present numerical example cases for a nonlinear two–DOF model with geometric nonlinearities and uncertainties. The nonlinear two–degrees–of–freedom model shows Fig.1 is chosen due to its simplicity and to better understand the effects of uncertainties of various physical parameters. The equations of motion take the following form for this system:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{Bmatrix} F_{nl1} \\ F_{nl2} \end{Bmatrix}$$
$$= \begin{Bmatrix} F_{11}\cos\omega_1 t + F_{12}\sin\omega_2 t \\ F_{21}\cos\omega_2 t + F_{22}\sin\omega_2 t \end{Bmatrix}$$

The cubic polynomial non linearity given by: $F_{nl1} = k_{nl1}(u_1)^3 - k_{nl2}(u_2 - u_1)^3$ and $F_{nl2} = k_{nl2}(u_2 - u_1)^3$. The values of the physical parameters are given in Table 1.

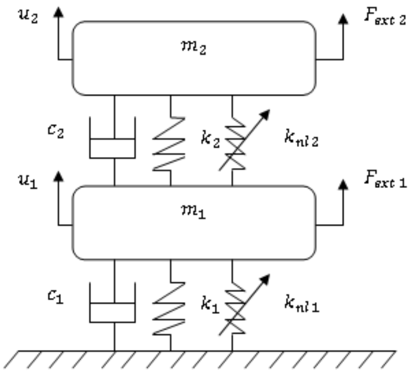

Figure 1: Two–DOF model with nonlinearities.

Table 1: Parameters values.

Parameters	m_1	m_2	c_1	c_2	k_1	k_2	k_{nl1}	k_{nl2}	F_{11}	F_{12}	F_{21}	F_{22}	v_m
Values	1	1	0.25	0	1	1	0	$0{,}1k_2$	1	0	0	0	2%

In this study, it has been chosen to investigate the effects of uncertainties by considering mass uncertain parameters. The mass parameter is supposed to be a random variable and defined as follows: $\widetilde{m} = \overline{m}(1 + \sigma_m v_m)$ with σ_m is a zero mean value Gaussian random variable. $\overline{m} = m_1 = m_2$ is the mean value v_m is the standard deviation of this parameter. Firstly, the mean and variance of the magnitude of frequency response have been computed by coupled HBM in order 1 and PCE in order 2 method.The obtained results are compared with those given by the direct Monte Carlo simulation 700 simulations. The obtained results are plotted in Figures 2,3,4 and 5 correspond respectively to mean and variance of DOF (1) and DOF (2) and they are a perfect agreement between PCE method and MCS.

4. Conclusion

In this study, the approach to calculating the random non linear dynamic response is presented, the classical methods require a considerable cost both in terms of computation time and data storage. The proposed method coupling a polynomial chaos expansion and harmonic balance method offers optimal cost then it is a powerful technique for non linear structures. Future work consists of applying method for complex structures in the mechanical engineering field.

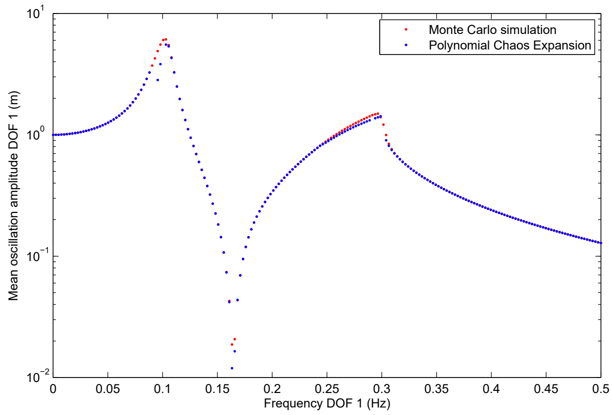

Figure 2: Mean frequency response of DOF (1).

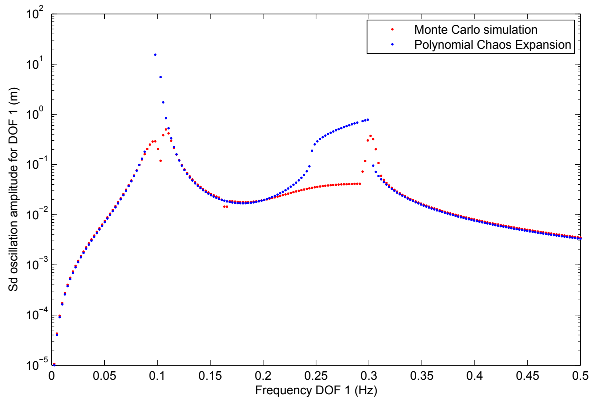

Figure 3: SD frequency response of DOF (1).

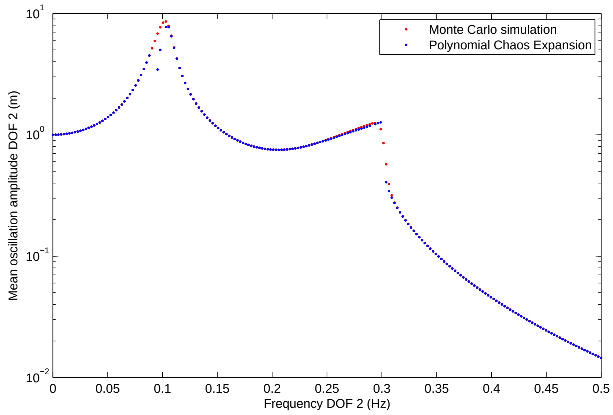

Figure 4: Mean frequency response of DOF (2).

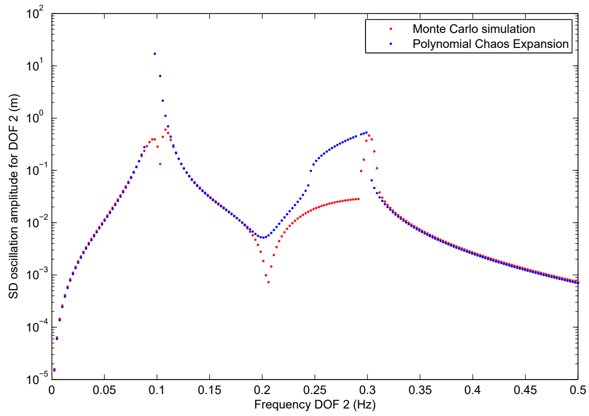

Figure 5: SD frequency response of DOF (2).

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