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MODELLING THE BOWED STRING

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When a string player presses sufficiently hard with his bow, the musical note gives way either to the familiar raucous 'beginner's noise' or, if the bow arm is steady enough, to more or less stable oscillations at pitches much lower than the fundamental string pitch. The reason why some such catastrophic breakdown must occur was made clear by Schelleng in his penetrating paper on the bowed string[1]. As Schelleng realised, maximum bow force in practice is generally less than the force causing breakdown, and it is signalled by one of several less drastic phenomena.

Of these phenomena, two are particularly important. The first is a very slight deviation of pitch - almost always on the 'flat' side of the string tuning - as bow force increases. This flattening is easily demonstrated with the bow a moderate distance from the bridge at a low bow speed[2]. When flattening is audible, pitch is sensitive to bow force so that control of intonation becomes difficult. Players avoid this regime. The second phenomenon is the gradual buildup of noise accompanying the musical note, which is noticeable when trying to play more and more loudly near the bridge. This noisy regime is frequently used to deliberate musical effect, but the noise can reach an unacceptable level, depending on context. This imposes the other major limitation on bow force in practice.

The physical causes of neither phenomenon have been elucidated as far as we know, although mention of flattening in the scientific literature goes back at least as far as Raman[3](p135). We present a theoretical model of the bowed string which predicts the flattening effect. The model also describes within a self-consistent framework some of the effects of bow force upon waveform detail, vindicating ideas which were first put forward by Cremer and Lazarus[4,5] and further developed by Schelleng. Progress toward understanding the second phenomenon (the buildup of noise) is reported elsewhere [6]. Its major cause is a mechanism depending on the finite width of the ribbon of bow-hair in contact with the string.

When slight flattening occurs, observations show that the motion of the string remains qualitatively similar to the normal Helmholtz motion, with a single, somewhat rounded, 'corner' travelling back and forth along the string. Observations also show the expected drop in frequency, equivalent to a delay in the round-trip time of the corner. The delay can happen only at the bow, so we must study in detail the processes of capture and release of the string.

Previous studies of these processes yielded important insights into the changes in waveform as bow force is varied, but did not predict the flattening behaviour we have described. Cremer[5] followed the course of the Helmholtz corner, examining the detailed changes in its shape occurring at different points in the cycle and seeing how these various changes can reach equilibrium to give a precisely periodic motion. Schelleng[1] showed how Cremer's 'secondary waves' produce the pattern of 'ripples' observed in velocity waveforms. We extend their discussion in two stages. In the first instance we follow Cremer and neglect the secondary waves. We later give examples of computer solutions which take them fully into account, and in the process simulate Schelleng's ripples. We begin with a re-examination of exactly what happens at the bow; this will reveal a new phenomenon which appears to account for the flattening effect.

Suppose an ideal Helmholtz corner, as shown in figure 1a, leaves the bow travelling towards the bridge. By the time it returns to the bow it will have been inverted and somewhat rounded, as shown by the solid curve in figure 1b. For the purposes of clarity in the discussion it is best to begin by assuming a form of rounding which is symmetric in time, and which is the same for the two sections of the string (although with different delays, depending on the distances to bridge and nut). A model string with these properties will be called 'quasi-symmetric'; it has harmonic overtones, but damping which increases with frequency.

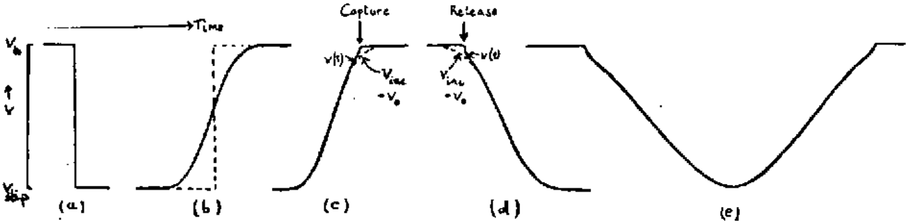

Figure 1. Successive waveforms for the Helmholtz corner, with dashed curves echoing the preceding stage in b-d (after Cremer and Lazarus[4]). In c and d, v_0 is a constant 'DC offset' arising from the fact that F(t) has a positive mean value.

We now ask what happens to the rounded corner as it passes the bow. In the absence of the bow, the velocity v(t) at this point would simply be equal to the incident velocity wave $v_{inc}(t)$ which we have plotted in figure 1b. The frictional force F(t) exerted by the bow produces an additional contribution to v(t), so that

$$v(t) = v_{inc}(t) + F(t)/2Z \qquad (1)$$

where Z is the relevant wave impedance[7], a constant property of the string. F and v are also related by the friction law: in the usual idealisation, F/F_b, where F_b is the normal component of bow force, has a functional dependence upon v of the kind sketched in figure 2[8]. From eq. (1) and figure 2 we can find F and v from the known v_{inc} at any given instant: they may be read off the friction curve as its intersection with the straight line of slope $2Z/F_b$ shown in figure 2. Cremer and Lazarus[4,5] discussed the case in which the bow force is sufficiently small that the slope of the friction curve is everywhere less than $2Z/F_b$, so that there is a simple one-to-one correspondence between the point of intersection and v_{inc}. This requires $F_b < F_{crit}$, where

$$F_{crit} = \frac{2Z}{\text{max. slope of graph in fig. 2}} \qquad (2)$$

Figure 1c shows $v_{inc}(t)$ (dotted) and the corresponding v(t) (solid) when F_b is well below F_{crit}. The wave transmitted past the bow towards the nut has the same shape v(t), and returns from the nut after being inverted and rounded again as in the dotted curve of figure 1d. This $v_{inc}(t)$ now produces the v(t) shown in the solid curve of figure 1d, and another round trip begins. The final curve (1e) shows the shape of the periodic solution which is closely approached after this chain of events has been repeated a few times. The period of the motion is precisely equal to the fundamental string period[9].

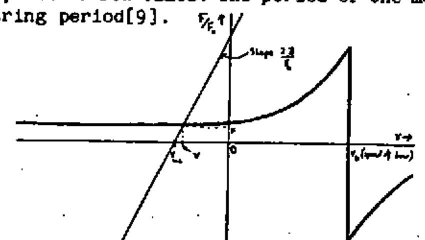

Figure 2

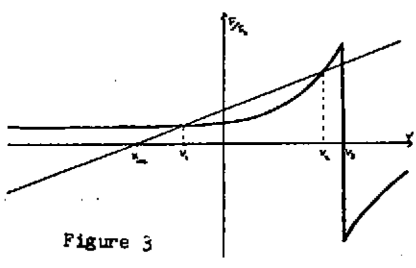

Figure 3

Now our proposed explanation of flattening hinges on a physically correct resolution of the ambiguity which arises when $F_b > F_{crit}$. The ambiguity, which was first pointed out by Friedlander[10], is illustrated in figure 3. There are now three intersections (v_1, v_2 and v_3) of the straight line and the friction curve, and we must decide which is appropriate at any given instant. It may be shown that the answer involves a kind of hysteresis, an essential difference between capture and release. The string chooses one of the two outer intersections (v_1 or v_3), according to the following rule:

> Slipping ($v=v_1$ in figure 3) will persist until v_{inc} reaches the value v_c shown in figure 4a; then capture occurs (v jumps to v_3). Sticking ($v=v_3$) will persist until v_{inc} reaches the value v_r shown in figure 4b; then release occurs (v jumps to v_1).

The parts of the friction curve shown dotted in figure 4 are traversed instantaneously in any model where (1) is regarded as exact. For a real string, these sections are presumably traversed in a finite but very small time.

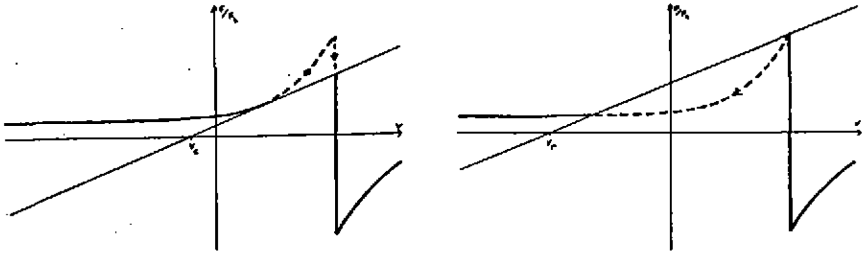

Figure 4 (a) Capture (b) Release

In figure 5, we plot the series of waveforms equivalent to those of figure 1 but with $F_b > F_{crit}$. Notice now that each time a rounded corner passes the bow, it has a 'bite' taken from it as the velocity jumps the relevant gap in figure 4, creating a discontinuity in v(t).

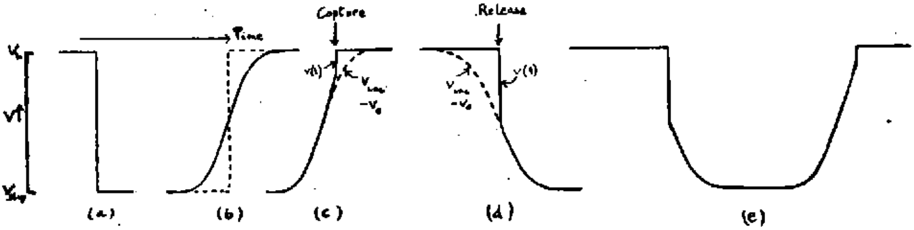

Figure 5. Same as figure 1, but with $F_b > F_{crit}$

Hysteresis means that the bite removed on release is bigger than that on capture. This produces a delay in the round-trip time of the Helmholtz corner, giving the flattening effect. The extent of flattening depends on the amount of hysteresis as well as on the amount of corner-rounding, so that the model predicts behaviour qualitatively similar to that commonly observed. For example, we now understand why notes high on a violin G string are particularly prone to flattening: corner-rounding is most drastic there.

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We now verify that the same effects occur in fully consistent computer simulations for the same quasi-symmetric bowed string. Two methods have been used, one using an integral-equation formulation [6] to find periodic solutions, the other simulating transient motion directly and waiting for the solution to settle down to periodicity. The methods have been shown to produce identical solutions given the same conditions. Examples of the resulting v(t) waveforms are shown in figures 6a, b and c, the first being for $F_b=F_{crit}$ and the others for $F_b>F_{crit}$. In Figure 6c the value of F_b is very close to that for breakdown to a raucous regime : the bite taken on release occupies the full height of the pulse[11]. That at capture is far smaller, and the note has flattened by about 50 cents (1/2 of a semitone). Details of the pulse shapes are different from those in figures 1e and 5e, especially for large F_b, because the secondary waves are now accounted for.

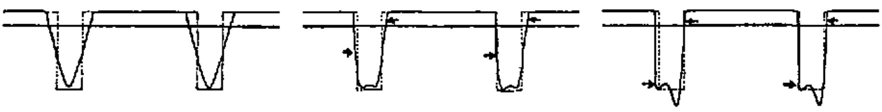

Figure 6. Velocity waveforms at the bow, from computer solutions for three values of F_b. The ideal Helmholtz pulse is shown dotted in each case, and the 'bites' are indicated by arrows.

References

(1) SCHELLENG, J.C. 1973 J.Acoust. Soc. Am. 53, 26-41 The bowed string and the player.

(2) There is then no significant amplitude-dependent sharpening, due simply to increased mean string tension, to confuse the issue.

(3) RAMAN, C.V. 1918 Indian Assoc. Cult. Sci. Bull. 15, 1-158 On the mechanical theory of vibrations of bowed strings, etc. .

(4) CREMER, L. AND LAZARUS, H. 1968 6th ICA Congress, Tokyo, N.2-3 .

(5) CREMER, L. 1974 Acustica 30, 119-136. Der Einfluss des 'Bogendrucks' auf die selbsterregten Schwingungen der gestrichenen Saite. English translation of an earlier version in Catgut Acoustical Society Newsletters 18 and 19, pp. 13-19 and 21-25.

(6) See MCINTYRE, M.E., SCHUMACHER, R.T. AND WOODHOUSE, J. 1977 Catgut Acoustical Society Newsletter 28, 27-31. New results on the bowed string. This abstract is a shortened version of this article. More detail of the work described here may be found in MCINTYRE, M.E. and WOODHOUSE, J. Fundamentals of bowed string dynamics, and SCHUMACHER, R.T. Self-sustained oscillations of the bowed string. (Both to appear in Acustica in 1978).

(7) For an ideal string Z is the usual wave impedance, defined as (string tension)/(wave speed). However, relation (1) holds even if torsional string motion is allowed, provided we redefine Z to have a smaller, but still constant, value[1], and take v(t) to be the velocity of the string surface.

(8) see CREMER, L. 1971 Nachr. Akad. Wiss. Göttingen : II Math. Physik. Kl. 12, 223-259. Die Geige aus der Sicht des Physikers.

(9) In Cremer's detailed analysis[5] he obtained a slight sharpening, attributable entirely to different terminations at the two ends of the string. We are using a quasi-symmetric string model to separate hysteresis-induced flattening from such additional effects, although real strings are of course not quasi-symmetric.

(10) FRIEDLANDER, F.G. 1953 Proc. Camb. Phil. Soc. 49, 3, 516. On the oscillations of the bowed string.

(11) c.f. equation (2) and footnote 10 of Schelleng[1].