Proceedings of The Institute of Acoustics

ON CONSERVABLE WAVE PROPERTIES FOR GENERAL ACOUSTIC DISTURBANCES.

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Introduction

In the theoretical description of waves propagating through moving media, it is desirable to have a measure of wave 'activity' which is not only a wave property, i.e. can be evaluated from linearised theory, but which satisfies a conservation equation when viscosity and other irreversible processes are neglected. We then have a convenient way of following the waves as they propagate from place to place (by evaluating the density and flux appearing in the equation), or as they are generated or dissipated (by evaluating the source or sink terms involving irreversible processes). This is not to be confused with the separate question of calculating the total transport of energy or momentum brought about by the waves.[1,2]

Suitable conservable quantities are well known in special cases, e.g. Blokhintsev's 'energy' density for the case of geometrical acoustics and irrotational, homentropic steady flow. Blokhintsev's restriction to geometrical acoustics was removed by Cantrell & Hart[3], while Whitham[4] and Bretherton & Garrett[5] retained the geometrical-acoustics assumption but allowed very much more general mean flows (and types of wave disturbance). For general mean flows without the geometrical-acoustics assumption Lighthill[3] has commented on "how difficult the position is".

However, it has recently become clear[6] (as a spin-off from work on waves in stratified, rotating fluids) how all the conservation relations in question arise as special cases of a yet more general relation (eq.5 below) arising from a synthesis of ideas from 'classical field theory' (the energy-momentum tensor) with the more recent work of Eckart[7], Hayes[8], Dewar[9] and Bretherton.[10] The relation in question is applicable to acoustic disturbances on arbitrary (unsteady, rotational, heterentropic, rapidly-varying) mean flows. The relationship with Whitham's[4] geometric-acoustics results has been discussed at length by Hayes[8]; here we indicate how simply the basic exact result, conservation of 'generalised wave-action', can be derived by elementary means, not involving variational principles. (For more details see reference 6.)

Description of the acoustic disturbance in terms of particle displacements

By far the simplest theoretical structure for general waves in fluid flows is obtained when we use the particle-displacement field $\boldsymbol{\xi}(\mathbf{x},t)$ as the fundamental disturbance variable.[6] (At root, the reason is that this gives the simplest way of formulating Hamilton's variational principle for the disturbance. It is ironical that a well-known attempt to express Hamilton's principle in purely Eulerian terms, using Clebsch potentials, led to the unintended reappearance of the Lagrangian description in the form of the celebrated 'Lin constraint'[11].)

Let $\underset{\sim}{u}'(\underset{\sim}{x},t)$ and $\bar{\underset{\sim}{u}}(\underset{\sim}{x},t)$ be the acoustic disturbance velocity and mean flow velocity respectively. Correct to $O(a)$, where a is a measure of wave amplitude, $\underset{\sim}{\xi}$ is defined to satisfy

$$\bar{D}_t\underset{\sim}{\xi} = \underset{\sim}{u}' + \underset{\sim}{\xi}.\nabla\bar{\underset{\sim}{u}} \equiv \underset{\sim}{u}^{\ell} \quad \text{(say)} \tag{1}$$

(the Lagrangian disturbance velocity), together with suitable initial conditions, where $\bar{D}_t \equiv \partial/\partial t + \bar{\underset{\sim}{u}}.\nabla$. We also have that the mean value of $\underset{\sim}{\xi}$ vanishes:

$$\overline{\underset{\sim}{\xi}(\underset{\sim}{x},t)} = 0, \tag{2}$$

and that

$$\bar{\rho}\,\nabla.\underset{\sim}{\xi} = -(\rho' + \underset{\sim}{\xi}.\nabla\bar{\rho}) \equiv -\rho^{\ell} , \tag{3}$$

again correct to $O(a)$, where $\bar{\rho}$ and ρ' are Eulerian mean and disturbance densities. (A corresponding definition for <u>nonlinear</u> disturbances can be given[12], but this will not be gone into here.)

<u>The basic conservable quantity A</u>.

To attain maximum generality we define the Eulerian averaging operator $\overline{(\ )}$ as an ensemble average, over an ensemble of wave solutions distinguished by a smoothly varying parameter α . In a stochastic problem α would range over a 'sample space'; but random waves are merely one possible case. In a deterministic problem in which the mean flow is steady, for example, we can generate a suitable ensemble by simply identifying α with time t. Then $\overline{(\ )}$ is equivalent to a time average. Quite generally, we have the basic property

$$\overline{\{\partial(\ )/\partial\alpha\}} = \partial\overline{(\ )}/\partial\alpha = 0 \tag{4}$$

whenever the ensemble of disturbance fields depends differentiably upon α, which we shall take to be the case.

The usual equation of acoustic energy (which as is well known is <u>not</u> a conservation equation when mean flow is significant) is obtained by dotting the linearised momentum equation with $\bar{\rho}\,\underset{\sim}{u}'$. If instead we take its dot product with the derivative $\bar{\rho}\partial\underset{\sim}{\xi}/\partial\alpha$ and average, there results [1,6], after some manipulation involving (4),

$$\boxed{\frac{\partial A}{\partial t} + \nabla.\underset{\approx}{B} = \bar{\rho}\overline{\frac{\partial \underset{\sim}{\xi}}{\partial\alpha} . \underset{\sim}{X}'} + \bar{s}\overline{\frac{\partial p^{\ell}}{\partial\alpha} S^{\ell}} + O(a^3)} \qquad (5)$$

where the right-hand side is zero for conservative motion, and

$$A = \bar{\rho}\overline{\underset{\sim}{u}^{\ell} . \partial\underset{\sim}{\xi}/\partial\alpha}$$

$$\underset{\approx}{B} = \bar{\underset{\sim}{u}}A + \overline{p'\partial\underset{\sim}{\xi}/\partial\alpha}.$$

The first term on the right is associated with a fluctuating body force $\underset{\sim}{X}'$ on the right of the momentum equation, which can represent either a viscous force per unit mass, or a given distribution of acoustic dipole strength. The second term on the right involves the Lagrangian disturbance pressure $p^{\ell} = p' + \underset{\sim}{\xi}.\nabla\bar{p}$, and the Lagrangian disturbance entropy $S^{\ell} = S' + \underset{\sim}{\xi}.\nabla\bar{S}$, which is evidently zero for adiabatic motion. The coefficient $\bar{s}$ is defined by $\bar{s} = (\partial \ln\bar{\rho}/\partial\bar{S})_{\bar{p}}$. No approximations whatever have been made, apart from that of small disturbance amplitude a; and even that can be removed if we define $\underset{\sim}{\xi}$ for finite-amplitude disturbances in a suitable way[6,12]. A may be called the generalized <u>wave-action</u>.

The reduction of this result to the more special results of Blokhintsev, Bretherton & Garrett[5] (in which α is replaced by phase shift[8]) and Cantrell & Hart[3], is indicated in reference 6. The relation to Hamilton's principle and the energy-momentum tensor is also discussed there. It turns out that the formulation in terms of $\underset{\sim}{\xi}$ also leads to a particularly powerful way of describing the back effect on the mean flow giving rise to such phenomena as radiation stress and acoustic streaming.[12]

<u>Concluding remarks</u>

It appears that the basic conservation relation (5) (with α replaced by t in the commonly important case of steady mean flow) may provide a generally useful answer to the long-standing problem of finding a conservable wave property for sound waves on a mean flow which can be rotational and heterentropic. For instance in the case of shear flow $\bar{u}(y,z)$ in a duct, past an arbitrary obstacle, constriction, or enlargement, eq.(5) may yield a way of expressing the acoustic scattering matrix in unitary (and therefore more compact) form.[13] The use of Clebsch potentials, which appears to be the only alternative in this situation, presents difficulties of the kind found in reference 13, eq.(14). However, to show that (5) is not subject to similar difficulties it must be demonstrated that

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$\xi(\underline{x},t)$ contains no contributions of the form func$\{x-\bar{u}(y,z)t,\ y,\ z\}$ (which make l.h.s.(1) zero)* and this has so far been done only for the two-dimensional case $\partial/\partial z = 0$ when the mean vorticity gradient $\bar{u}''(y)$ does not vanish.

On the question of physical interpretation, it should not be thought that equation (5), even for the case of steady mean flow, is a straightforward expression of conservation of energy and/or momentum. Strictly speaking, the derived quantities arising in place of A when ensemble averaging is replaced by time or space averaging are pseudoenergy and pseudomomentum (which are associated respectively with temporal and spatial symmetry of the mean flow), and not energy and momentum (which are associated with temporal or spatial symmetry of the total physical problem[14], and are not generally wave properties).

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