STRATEGIES FOR ACTIVELY CONTROLLING FLEXURAL WAVES ON AN EULER-BERNOULLI BEAM

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ABSTRACT

The active control of flexural waves propagating along an Euler-Bernoulli beam is considered. A theoretical model and computer simulations are presented to show how an array of forces can be applied to a beam to couple into and suppress a propagating flexural wave. The mechanism of wave reflection and absorption together with power flow is examined. The control strategies of minimizing secondary force effort and maximizing power absorption by the secondary force array are discussed for an infinite and a finite beam.

INTRODUCTION

Practical engineering structures are often fabricated from a number of components held together by structural elements that may be modelled as one dimensional waveguides. Examples of such structures are truss beams, antenna booms and struts and tie-bars found in machinery installations in ships, submarines and helicopters. Active control of vibrations in these structures involves cancelling unwanted disturbances by deliberately adding secondary controlled disturbances. Rather than adopt global control of the system dynamics [1][2], another approach is to prevent the transmission of vibrational power between the structural components. For this type of local control of vibrations a modal model of the structure such as that described by Ewins [3] is inappropriate, and the system dynamics can better be described by a wave model [4][5][6]. In general, three wave types will be present in connecting elements between the structural components; flexural, or out-of-plane waves, longitudinal, or in-plane waves and torsional waves [7]. Local structural control of these waves can be achieved by placing actuators along the waveguides. The controlled secondary disturbances can be generated by generic actuators that can apply either a force, a moment, or a pair of moments in anti-phase (a moment-pair).

This paper restricts the investigation into how forces positioned in a secondary array couple into flexural waves propagating along a Euler-Bernoulli beam. If the control of flexural waves are understood, the behaviour of other wave types can be deduced by ignoring the near-field terms associated with flexural waves. The analysis is conducted in the frequency domain which is appropriate for deterministic disturbances, but this paper does not address how control of the secondary actuators can be implemented, i.e. how the magnitude and phase of the sinusoid fed to each actuator can be adjusted in practice.

Two control strategies are investigated; that of minimizing the secondary effort required to suppress a propagating flexural wave and that of maximizing the power absorbed by the secondary array. Nelson and Elliott [8] considered the strategy of minimizing secondary effort in the cancellation of noise in an acoustic enclosure, and this is applied here to the structural case. Redman-White *et al* [9] reported experiments in flexural wave control on an infinite beam using a control strategy which maximized power absorbed by two secondary forces. Nelson *et al* [10] and Elliott *et al* [11] have also considered this same strategy in the acoustic case, and Elliott concluded that in an enclosed sound field where the primary and secondary sources are well coupled, this control strategy results in an increase in the total power output into the enclosure. This paper shows that this control strategy, when applied to a finite beam also results in an increase in power input.

2. MODEL OF THE SECONDARY FORCE ARRAY

Consider an infinite Euler-Bernoulli beam with a secondary force array positioned as shown in figure 1. The displacement at any point along the beam can be described by the equation [4]:

$$w(x) = A_1 e^{kx} + A_2 e^{-kx} + A_3 e^{jkx} + A_4 e^{-jkx} + A_i e^{-jkx} \quad (1)$$

where the time dependence is suppressed for clarity and no energy loss in the beam is assumed. The flexural wave incident on the array is A_i, and the waves generated by the secondary force array are two near-field waves A_1 and A_2, and two propagating waves A_3 and A_4. The wave constants either side of the array, in the regions of the beam $x \le 0$ and $x \ge l_3$ can be described by the matrix equation:

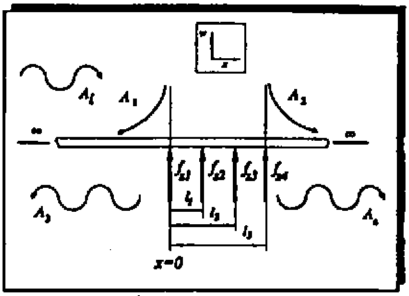

Figure 1 A Secondary Force Array on an Infinite Beam

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = -\frac{1}{4EIk^3} \begin{bmatrix} 1 & e^{-kl_1} & e^{-kl_2} & e^{-kl_3} \\ 1 & e^{kl_1} & e^{kl_2} & e^{kl_3} \\ j & je^{-jkl_1} & je^{-jkl_2} & je^{-jkl_3} \\ j & je^{jkl_1} & je^{jkl_2} & je^{jkl_3} \end{bmatrix} \begin{bmatrix} f_{s1} \\ f_{s2} \\ f_{s3} \\ f_{s4} \end{bmatrix} \quad (2)$$

where E is the modulus of elasticity, I is the second moment of area and k is the flexural wave number. This equation gives the amplitudes of the waves generated by the secondary forces either side of the array. The amplitudes of the waves within the array may be calculated by three similar relationships. These are:

for $0 \le x \le l_1$

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = -\frac{1}{4EIk^3} \begin{bmatrix} 0 & e^{-kl_1} & e^{-kl_2} & e^{-kl_3} \\ 1 & 0 & 0 & 0 \\ 0 & je^{-jkl_1} & je^{-jkl_2} & je^{-jkl_3} \\ j & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{s1} \\ f_{s2} \\ f_{s3} \\ f_{s4} \end{bmatrix} \quad (3)$$

for $l_1 \le x \le l_2$

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = -\frac{1}{4EIk^3} \begin{bmatrix} 0 & 0 & e^{-kl_2} & e^{-kl_3} \\ 1 & e^{kl_1} & 0 & 0 \\ 0 & 0 & je^{-jkl_2} & je^{-jkl_3} \\ j & je^{jkl_1} & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{s1} \\ f_{s2} \\ f_{s3} \\ f_{s4} \end{bmatrix} \quad (4)$$

for $l_2 \le x \le l_3$

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = -\frac{1}{4EIk^3} \begin{bmatrix} 0 & 0 & 0 & e^{-kl_3} \\ 1 & e^{kl_1} & e^{kl_2} & 0 \\ 0 & 0 & 0 & je^{-jkl_3} \\ j & je^{jkl_1} & je^{jkl_2} & 0 \end{bmatrix} \begin{bmatrix} f_{s1} \\ f_{s2} \\ f_{s3} \\ f_{s4} \end{bmatrix} \quad (5)$$

These are general equations and can be used for any number of secondary forces up to a maximum of four. Should greater than four secondary forces be applied, then the system of matrices can be extended with relative ease. The purpose of the secondary force array is to generate a set of waves such that at least the incident propagating wave is suppressed. If a single secondary force is applied, the incident wave is the only wave that can be suppressed. As the secondary force array generates four waves, there is a residual near-field wave and a propagating wave, upstream of the array, and a near-field wave downstream. When more than one force is applied, additional waves that are generated by the array can be suppressed; one wave for each additional force. It follows, therefore, that if four forces are applied, then all waves are suppressed. The non-dimensionalized displacement of the beam when one to four forces are applied for selected control strategies are shown in figures 2-4, where λ is a wavelength of the incident propagating wave. In figure 3 the residual waves are both near-field waves. In figure 4 the residual wave is the negative-going near-field wave.

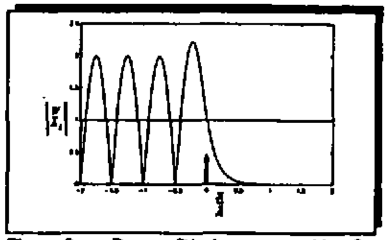

Figure 2 Beam Displacement with One Secondary Force Applied to Suppress A_4

To calculate the complex wave constants the magnitude and phase of the secondary forces are required. They can be determined from the equation:

$$\begin{bmatrix} f_{s1} \\ f_{s2} \\ f_{s3} \\ f_{s4} \end{bmatrix} = -4EIk^3 \begin{bmatrix} 1 & e^{-kl_1} & e^{-kl_2} & e^{-kl_3} \\ 1 & e^{kl_1} & e^{kl_2} & e^{kl_3} \\ j & je^{-jkl_1} & je^{-jkl_2} & je^{-jkl_3} \\ j & je^{jkl_1} & je^{jkl_2} & je^{jkl_3} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -A_i \end{bmatrix} \quad (6)$$

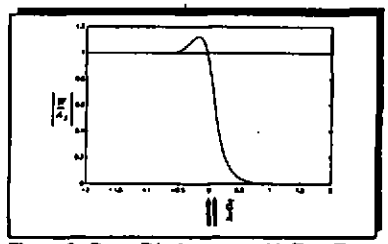

Figure 3 Beam Displacement with Two Forces Applied to suppress A_3 and A_4. Distance Between Secondary Forces = 0.1λ

If less than four secondary forces are applied then this equation can be partitioned accordingly. For example if only two secondary forces are applied and the propagating waves away from the array are suppressed the equation reduces to:

$$\begin{bmatrix} f_{s1} \\ f_{s2} \end{bmatrix} = -4EIk^3 \begin{bmatrix} j & je^{-jkl_1} \\ j & je^{jkl_1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -A_i \end{bmatrix} \quad (7)$$

which expands to give:

$$f_{s1} = \frac{4EIk^3 e^{-jkl}}{2\sin kl} A_i \quad (8)$$

and

$$f_{s2} = -\frac{4EIk^3}{2\sin kl} A_i \quad (9)$$

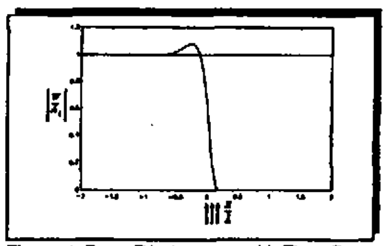

Figure 4 Beam Displacement with Three Forces Applied to Suppress A_2, A_3 and A_4. Distance Between each Force = 0.1λ

The similarity should be noted between these equations and those resulting from the use of two loudspeakers in a duct to absorb an incident acoustic wave [8]. Examination of equations (8) and (9) shows that the secondary forces required are infinite when the distance between the secondary forces equals an integer number of half-wavelengths of the incident propagating wave. This shows that it is not possible to suppress both an incident propagating wave and a negative-going propagating wave generated by the array at these frequencies.

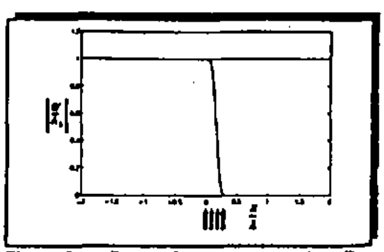

Figure 5 Beam Displacement with Four Secondary Forces Applied to Suppress all Waves. Distance Between each Secondary Force = 0.1λ

It is interesting to examine the secondary effort required to for the cases depicted in figures 2-4 as a function of frequency (distance between the secondary forces divided by a wavelength). The secondary effort required, (E), is defined as the sum of the squared secondary forces divided by the force required to generate the incident propagating wave, which is:

$$E = \frac{\sum_{n=1}^{N} |f_{sn}|^2}{|f_p|^2} \tag{10}$$

where $f_p = j4EIk^3 A_i$ and N is the number of forces. The effort is plotted in figure 6 and is quantified in the table for a distance of 0.1λ.

Number of Forces	1	2	3	4
Effort Required (E)	1	1.45	5.89	27.13

It can be seen that the effort increases with the number of forces, and provided the distance between the forces is less than a quarter of a wavelength, the effort required in all cases decreases with frequency. As the distance between the forces approaches zero or an integer number of half-wavelengths of the incident propagating wave, then the forces become prohibitively large. The minimum effort required in all cases reaches a minimum when the distance between the secondary forces is between 0.25λ and 0.3λ.

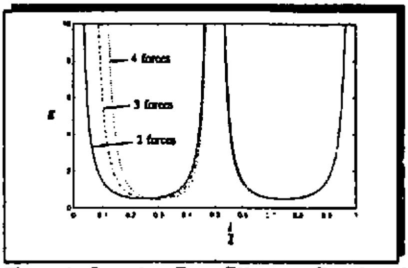

Figure 6 Secondary Force Effort as a Function of Distance between the Forces Compared to a Wavelength

3. ACTIVE CONTROL USING MINIMUM EFFORT

The control strategies discussed above involve reducing the far-field displacement downstream of the array to zero, and if more than one secondary force is used, to additionally suppress the remaining waves generated by the array. Another control strategy is to suppress the incident propagating wave using minimum secondary effort. This is a constrained optimization problem which is well documented in the literature, for example [12]. Consider an actuator array with N actuators positioned at $x = 0, l_2, \ldots l_N$ as shown in figure 7. The N secondary forces may be represented as a vector:

$$E_s^T = [f_{s1}(0) \;\; f_{s2}(l_2) \;\; f_{s3}(l_3) \;\ldots\; f_{sN}(l_N)] \tag{11}$$

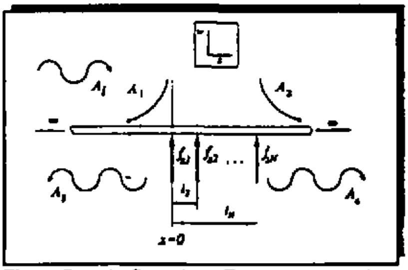

Figure 7 A Secondary Force Actuator Array Positioned on an Infinite Beam

and the positive-going propagating wave generated by the secondary force array is:

$$A_4 = Z\,E_s \tag{12}$$

where:

$$Z = -\frac{j}{4EIk^3}\left[1 \;\; e^{jkl_2} \;\; e^{jkl_3} \;\ldots\; e^{jkl_N}\right] \tag{13}$$

Now, the displacement of the beam in the far-field, downstream of the actuator array away from any near-field effects, and suppressing time dependence, is:

$$w(x) = A_4 + A_i \tag{14}$$

substituting for A_4 from equation (12) this becomes:

$$w(x) = \underline{Z}\underline{E}_s + A_i \qquad (15)$$

The object is to minimise the secondary effort, $\underline{E}_s^H \underline{E}_s$ subject to the constraint that the downstream displacement is zero, i.e. $\underline{Z}\underline{E}_s + A_i = 0$. Where ^{H} denotes the hermitian transpose, which is the transpose of the complex conjugate. The Lagrange function (cost function) is thus [8]:

$$J = \underline{E}_s^H \underline{E}_s + \mu^*(\underline{Z}\underline{E}_s + A_i) + \mu(\underline{Z}\underline{E}_s + A_i)^* \qquad (16)$$

where, μ is the Lagrange multiplier. The problem is now reformulated with the constraint incorporated in the cost function. This new unconstrained function now has to be minimised. Noting that equation (17) is of hermitian quadratic form in both $\underline{E}_s$ and μ, the complex derivatives of J with respect to both the real and imaginary parts of these quantities are:

$$\frac{\partial J}{\partial \mu_R} + j\frac{\partial J}{\mu_I} = 2(\underline{Z}\underline{E}_s + A_i) \qquad (17)$$

$$\frac{\partial J}{\partial \underline{E}_{sR}} + j\frac{\partial J}{\partial \underline{E}_{sI}} = 2\underline{E}_s + 2\mu \underline{Z}^H \qquad (18)$$

Where the suffices R and I denote real and imaginary parts. The minimum value of $\underline{E}_s$ within the constraint, is given by setting both equations (17) and (18) to zero. This results in the optimum secondary force vector

$$\underline{E}_{so} = -\frac{\underline{Z}^H A_i}{\underline{Z}\underline{Z}^H} \qquad (19)$$

(It should be noted that a more general problem of this type occurs in the active control of a sound field where there are fewer microphones (error sensors) than loudspeakers (secondary sources). In this case the Lagrange Multiplier turns out to be a vector and the reader is referred to reference [8] for a full explanation on how to deal with a problem of this type).

Now, substituting for $\underline{Z}$ from equation (13), equation (19) becomes:

$$\underline{E}_{so} = -\frac{j4EIk^3}{N}\left[1\ e^{-jkl_2}\ e^{-jkl_3}\ \ldots\ e^{-jkl_n}\right]^T A_i \qquad (20)$$

The force required to generate a propagating wave with amplitude $|A_i|$ is $f_p = j4EIk^3 A_i$. Hence substituting this into equation (20) and computing the modulus, gives the expression for the sum of the squared minimum secondary forces:

$$\sum_{n=1}^{N} |F_{sn}|^2 = \underline{E}_{so}^H \underline{E}_{so} = \frac{|f_p|^2}{N} \qquad (21)$$

This rather surprising result shows that the minimum secondary force vector required to suppress an incident travelling wave is independent of the distance between the secondary forces and the secondary effort *reduces* as the number of secondary forces *N increases*. The displacement of a beam with this strategy implemented with four secondary forces applied, and a distance between the secondary forces of 0.1λ, is shown in figure 8. Upstream of the secondary force there is a partial standing wave, which implies the upstream propagating wave is smaller than the incident wave, i.e., the array has partially absorbed the incident wave. The reflected wave (A_r) is given by:

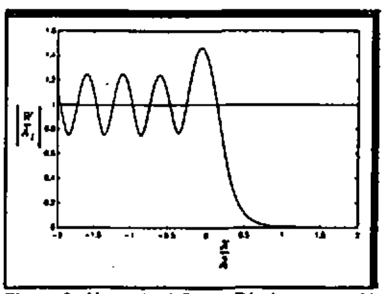

Figure 8 Normalized Beam Displacement with Four Secondary Forces Applied. Distance Between Secondary Forces = 0.1λ

$$A_3 = -\frac{j}{4EIk^3}\left[1 \quad e^{-jkl_1} \quad e^{-jkl_2} \dots e^{-jkl_N}\right]\begin{bmatrix} f_{s1} \\ f_{s2} \\ f_{s3} \\ \cdot \\ \cdot \\ \cdot \\ f_{sN} \end{bmatrix} \qquad (22)$$

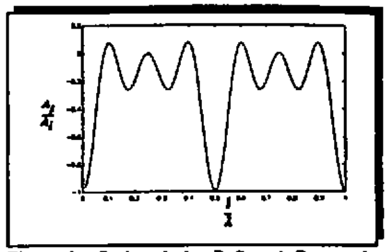

Figure 9 Ratio of the Reflected Propagating Wave to the Incident Wave

Substituting the minimum secondary force vector from equation (20) gives the expression for the ratio of the reflected propagating wave to the incident propagating wave:

$$\frac{A_3}{A_i} = -\frac{1}{N}\left[1 + e^{-j2kl_1} + e^{-j2kl_2} + \dots e^{-j2kl_N}\right] \qquad (23)$$

This is plotted for four secondary forces in figure 9, where l is the distance between each of the secondary forces. It can be seen that the magnitude of the A_3 wave is dependent upon the distances between the secondary forces compared to a wavelength, but for $0.1 < l/\lambda < 0.4$, the reflected wave has an amplitude which is less than 30% of the incident wave amplitude.

4. MAXIMUM POWER ABSORPTION

A localised control strategy described by Redman-White *et al* [9] for Euler-Bernoulli beams and Elliott *et al* [11] for an acoustic enclosure, is to maximize the power absorbed by a secondary array. On a lossless infinite beam with two secondary forces applied, and at frequencies where interaction between primary and secondary near-fields is negligible, this strategy is equivalent to suppressing the incident propagating wave and the negative-going propagating wave generated by the array. Consider the beam shown in figure 10. The vector of secondary forces which maximizes the power absorbed by these forces is given by:

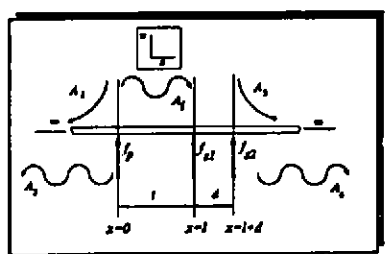

Figure 10 A Beam with a Primary Force Applied and a Secondary Array with Two Forces

$$\underline{f}_{ss} = -\frac{1}{2}\underline{R}_{ss}^{-1}\underline{M}_{sp}f_p \qquad (24)$$

where $\underline{R}_{ss}$ is the real part of the mobility matrix of the secondary force array, and $\underline{M}_{sp}$ is the transfer mobility matrix between the primary force and the velocities at the secondary force positions. This can be re-cast in terms of displacements, which enables an analysis to be performed which is consistent with the control strategies described above. This is:

$$\underline{f}_{ss} = \frac{j}{2}\underline{X}_{ss}^{-1}\underline{\alpha}_{sp}f_p \qquad (25)$$

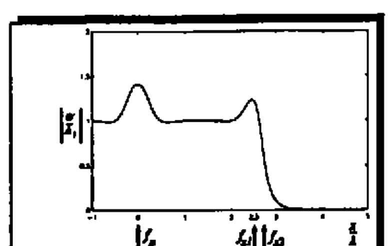

Figure 11 Normalized Beam Displacement with Two Secondary Forces Configured to Absorb Maximum Power

where $\underline{X}_{ss}$ is the imaginary part of the receptance matrix of the secondary force array, and $\underline{\alpha}_{sp}$ is the transfer receptance matrix between the primary force and the secondary force positions. The calculated secondary forces from equation (25) can be substituted into equations (1)-(5), which can be simply modified to account for a primary force, to determine the beams displacement. A plot of displacement with a distance between the primary and secondary forces of 2.5λ and a distance between the secondary forces of 0.25λ is shown in figure 11. It can be seen that this strategy involves absorbing the incident wave, as no negative-going propagating wave is generated by the array. The time averaged power input by the forces is given by:

$$P = -\frac{\omega}{2}\,\mathrm{Im}\{f^{*}\,w\} \qquad (26)$$

substituting for the calculated forces and displacements, and neglecting near-field waves, gives the power input by the primary force:

$$P_p = \frac{\omega|f_p|^2}{8EIk^3} \qquad (27)$$

which is the same as the power input by a single point harmonic force on an infinite beam. This shows that the secondary force array has no effect on the power input by the primary force. The power input by the secondary forces can be similarly calculated to give:

$$P_{s1} = \frac{-\omega|f_p|^2}{16EIk^3}\left\{1 + \frac{e^{-kd}}{2\sin kd}\right\} \qquad (28)$$

and

$$P_{s2} = \frac{-\omega|f_p|^2}{16EIk^3}\left\{\frac{e^{-kd}}{2\sin kd}\right\} \qquad (29)$$

Examination of these equations shows that only f_{s1} absorbs power, and if the near-field interaction between the secondary forces can be neglected, it absorbs half the power that is input into the beam by the primary force, that is, all the power incident upon it, carried in the propagating flexural wave. In the more general case when near-field effects are considered then f_{s2} actually supplies power in a near-field wave which interacts with the near-field wave of f_{s1}, and is absorbed by f_{s1}. The magnitude of power input and absorbed by all the forces are shown in figure 12.

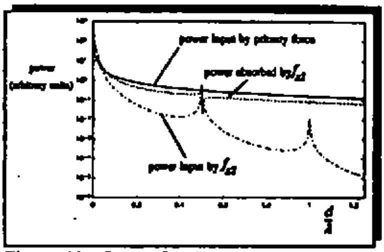

Figure 12 Power Input and Absorbed by the Primary and Secondary Forces

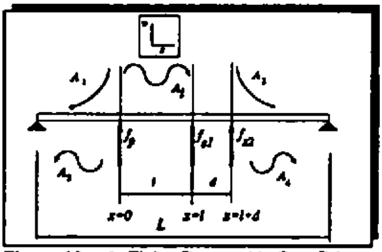

Figure 13 A Finite Beam with One Primary Force and Two Secondary Forces Applied

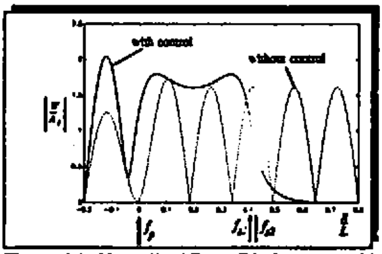

Figure 14 Normalized Beam Displacement with Two Secondary Forces Configured to Suppress the Incident Propagating Wave

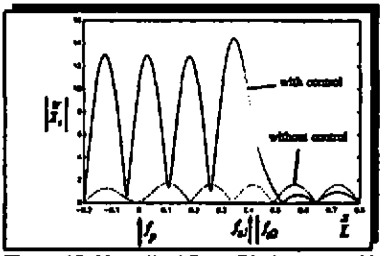

Figure 15 Normalized Beam Displacement with Two Secondary Forces Configured to Absorb Maximum Power

This situation changes considerably when a finite beam is considered, such as that in figure 13. The strategy of suppressing an incident propagating wave remains effective, as shown in figure 14, but the strategy of the secondary array absorbing maximum power turns out to be a poor control strategy to adopt. The reason is that with boundaries in place, it is possible for the secondary array to influence the conditions at the primary force position, such that more power is supplied to the beam when the control strategy is implemented. The result is a large displacement of the beam upstream of the secondary array. Figure 15 shows the displacement of the beam when the secondary forces are adjusted to maximize power absorbed. For the simulations presented, 1% damping is included in the beam model, by way of a complex modulus of elasticity. This is necessary to enable $\underline{X}_{ss}$ to be inverted. The effects of this strategy are self evident, with large increases in beam displacement with control. It should be noted that the frequency of the incident wave is such that the beam is not resonant ($L/\lambda = 3.25$), and the magnitude of the standing wave with control is frequency dependent. The power input by the primary force before and after control for both strategies is plotted in figures 16 and 17. Examination of these figures shows clearly that the strategy to absorb maximum power causes the primary force to input more power with control.

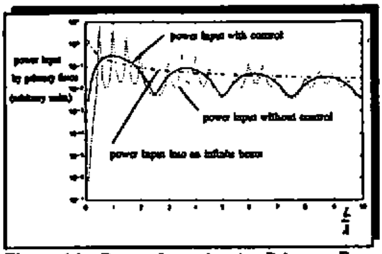

Figure 16 Power Input by the Primary Force with two Secondary forces Configured to Suppress the Incident and Reflected Propagating Waves

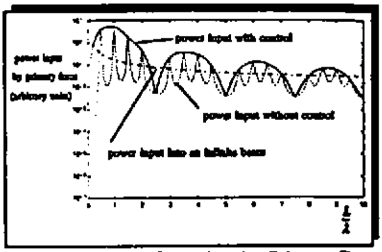

Figure 17 Power Input by the Primary Force with Two Secondary Forces Configured to Absorb Maximum Power

5. CONCLUSIONS

A model of a secondary force array which couples into flexural waves on an Euler-Bernoulli has been proposed, together with examples of how the secondary array can be used to suppress an incident propagating wave and some or all of the waves generated by the array. Two additional control strategies have been considered; one which minimizes the secondary effort required to suppress a propagating wave and one which absorbs maximum vibrational power. It was shown that although the strategy of maximizing the power absorption of the secondary force array is a viable control strategy on an infinite beam, it is not on a finite beam, as it causes more power to be input by the primary force. Maximizing secondary power absorption would thus not appear to be a viable control strategy on a finite beam if global control of vibration is required.

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