ROUGH SURFACE SCATTERING AND THE INVERSE PROBLEM: LOW GRAZING ANGLES

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1 INTRODUCTION

The problem of acoustic scattering at low grazing angles is of importance to studies of surface and bottom reverberation; but techniques developed for acoustics applications will find application in a wide range of other disciplines: such as the design of noise reducing tiles and coatings and clutter for sea-borne surveillance radars.

This paper considers the results that can be derived using the parabolic equation method, which is applicable to low angle scattering, and briefly mentions possible extensions to this technique that have wider applicability.

The paper begins with a formulation of the low angle scattering problem, and then goes on to consider effective reflection coefficients and the angular distribution of intensity. Numerical solutions to the inverse problem are then described; and finally extensions of the method to larger angles are presented.

2 FORMULATION OF PROBLEM

Consider a time-harmonic wavefield p scattered from a one-dimensional rough surface $h(x)$, with a pressure release boundary condition. The wavefield propagates with wavenumber k, and is governed by the wave equation $(\nabla^2 + k^2)p = 0$. Angles of incidence and scattering will be assumed to be small with respect to the surface. The coordinate axes are x and z, where x is the horizontal $x \geq 0$, and z the vertical directed out of the medium.

The field has a slowly-varying part $\psi_{tot}(x,z) = p(x,z)\exp(-ikx)$. Incident and scattered components ψ_i and ψ_s are defined similarly, so that $\psi_{tot} = \psi_i + \psi_s$. This field then approximately obeys the parabolic wave equation

$$2ik\psi_x + \psi_{zz} = 0 \tag{1}$$

3 GOVERNING EQUATIONS

The field is related to the surface by the following equations eg [1, 2]:

$$\psi_i(\mathbf{r}) = -\int_0^x G(\mathbf{r};\mathbf{r}')\frac{\partial \psi(\mathbf{r}')}{\partial z}dx' \tag{2}$$

where both $r=(x,h(x))$, $r'=(x'h(x'))$ lie on the surface: and

$$\psi_s(\mathbf{r}) = \int_0^x G(\mathbf{r};\mathbf{r}')\frac{\partial\psi(\mathbf{r}')}{\partial z}dx' \tag{3}$$

where $\mathbf{r}'$ is again on the surface and $\mathbf{r}$ is now an arbitary point in the medium. Here G is the parabolic form of the Green's function in two dimensions given by

$$G(x,z;x',z')\begin{cases} =\alpha\sqrt{\dfrac{1}{x-x'}}\exp\left[\dfrac{ik(z-z')^2}{2(x-x')}\right] & \text{for } x'<x \\ =0 & \text{otherwise}\end{cases} \tag{4}$$

where $\alpha=\frac{1}{2}\sqrt{i/2\pi k}$. This form gives rise to the finite upper limit of integration in (2) and (3), which holds provided the angles of incidence and scattering are fairly small with respect to the x-direction.

The main analytical and numerical problem is the inversion of (2) to give the "induced source" $\partial\psi/\partial z$ at the surface.

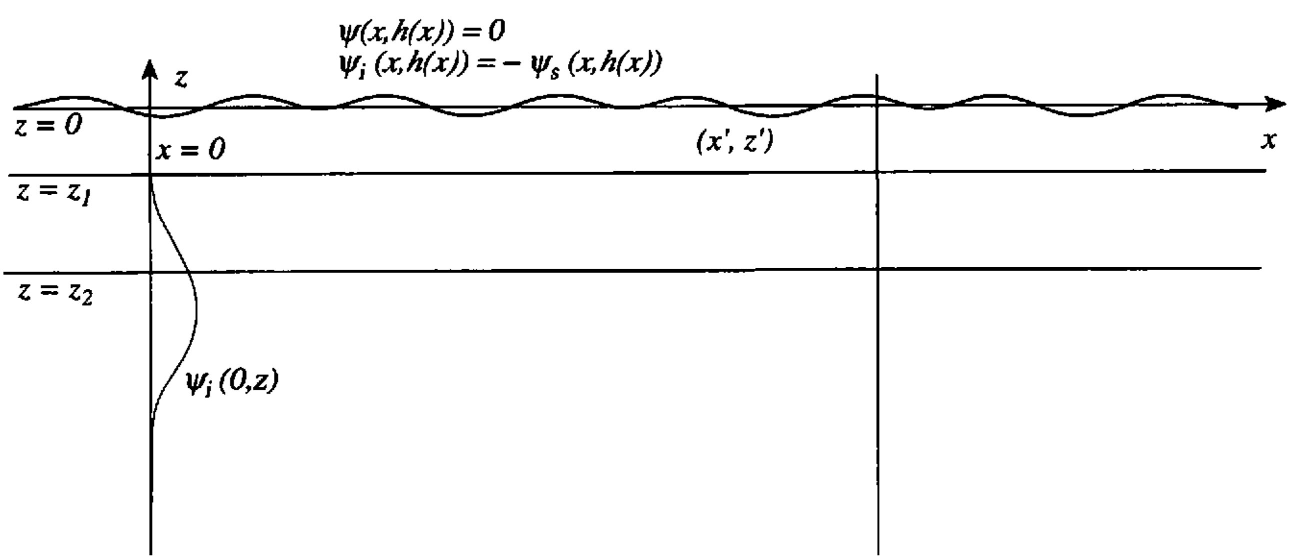

Figure 1: Diagram defining the symbols used in this paper

4 EFFECTIVE REFLECTION COEFFICIENTS AND ANGULAR DISTRIBUTION OF INTENSITY

We consider the dependence of the scattered field on the surface statistics and the grazing angle. In particular we seek effective reflection coefficients, and the angular distribution of intensity, and examine how they change as the incident angle approaches zero.

At low grazing angles most analytical methods such as the usual perturbation method and the Kirchhoff approximation break down (with some exceptions - eg [3, 4])

An adaption of the parabolic equation method provides a tractable approach to the problem.

5 EFFECTIVE REFLECTION COEFFICIENTS

Suppose a plane wave is incident upon a statistically stationary rough surface. Then the coherent scattered field $<\psi_s>$ is a specularly-reflected plane wave. This quantity is given by a effective or 'averaged' reflection coefficient $\mathbf{T}_\theta$ which characterises the rough surface and depends on angle of incidence θ. It is well-known (for a pressure-release surface) that as the grazing angle ϕ tends to zero, the surface becomes perfectly-reflecting, so that $\mathbf{T}_\theta$ approaches its smooth-surface form.

6 IMAGE PROPERTY

From the above observation the following invariance holds for an arbitary source of illumination: *The coherent field remains unchanged if the source and observation point are displaced by equal distances in opposite directions.* This holds for an arbitrary source S because, by specular reflection, it applies to every plane wave component of S.

7 MEAN FIELD

Scattered field near surface:

We first find the field along a plane: Assume slight roughness so that there is a plane at z_1 close to every point on the surface $h(x)$. Expand the scattered field $\psi_s(x,z_1)$ along this plane about $h(x)$ to second order in (z_1-h). We thus obtain

$$\psi_s(x,z_1) \cong \psi_s(x,h) + [z_1-h]\frac{\partial\psi_s(x,h)}{\partial z} + \frac{1}{2}[z_1-h]^2\frac{\partial^2\psi_s(x,h)}{\partial z^2} \tag{5}$$

In this expression the z- derivatives of ψ_s at the surface can be written in terms of derivatives of the known incident field, and the governing parabolic integral equation. After neglecting higher order terms we can invert this analytically, to obtain an explicit expression for the field ψ_s in terms of the surface.

Average:

The scattered field is now in a form which is easily averaged. We obtain

$$\left\langle \psi_s^\theta(x,z_1)\right\rangle \cong -e^{ik[\sin\theta-1]x}(1+T_\theta) \tag{6}$$

Here $(1+T_\theta)$ is the effective reflection coefficient which we seek; T_θ is given by

$$T_\theta = \frac{k^2}{\pi\alpha}\cos\theta\left\{\int_o^\infty e\frac{-iks\xi}{\sqrt{\xi}}\left[\rho(\varepsilon)S - \frac{1}{ik}\frac{d\rho(\xi)}{d\xi}\right]d\xi\right\} \tag{7}$$

where ρ is the surface autocorrelation function. For example, for a *fractal surface* this becomes

$$T_\theta = \frac{k^2}{\pi\alpha}\sigma^2\cos\theta\frac{S+1/ikL}{\sqrt{ikS+1/L}} \tag{8}$$

8 ANGULAR DISTRIBUTION OF INTENSITY

Using the explicit expression (5) for the scattered field along the plane it is possible to find the higher moments on the plane, in closed form. The moments of the field anywhere in the medium can also be found, but in general this can only be done in terms of multiple integrals which cannot easily be written in closed form.

However, it can be shown that the second moment does not change with propagation away from a fixed plane (apart from deterministic phase change due to distance). Thus the intensity distribution in the medium can be found in terms of its value along the fixed near-surface plane.

In order to form the higher moments of the wavefield we take powers and polynomials of the above expression for the scattered field, average, and truncate at second order. The scattered field has the form $(1+a\varepsilon(x)+b\varepsilon^2(x))$ where ε is a small random function. Consider for simplicity the expression

$$P_n = \left(1+[\varepsilon_1+\varepsilon_2]\right)^n \tag{9}$$

where ε_j is random, of order ε^j for j = 1,2 and all odd-order terms (eg ε_1 and $\varepsilon_1\varepsilon_2$) have mean zero. This has binomial expansion

$$P_n = 1+n(\varepsilon_1+\varepsilon_2)+C_{n,2}(\varepsilon_1+\varepsilon_2)^2+\ldots+(\varepsilon_1+\varepsilon_2)^n \tag{10}$$

Expanding further, truncating at second order, and averaging yields

$$\langle P_n\rangle \cong 1+n<\varepsilon_2>+C_{n,2}<\varepsilon_1^2> \tag{11}$$

The error thus incurred clearly diverges for large n since the coefficients $C_{n,j}$ grow exponentially with n. The approximation is reasonable provided $n^2\varepsilon^4 << 1$. Once the quantities required for the effective reflection coefficients have been obtained, the higher moments near the surface are relatively straightforward to evaluate from $<P_n>$ above.

9 NUMERICAL TREATMENT

The principal advantage of the parabolic equation description becomes clear when the system is discretized. Due to the upper limit of integration, in the resulting matrix equation the matrix is lower triangular, and inversion is carried out by Gaussian elimination, which is highly efficient. This requires $0(N^2)$ operations rather than $0(N^3)$ needed to invert a full matrix in general.

More explicitly: The region of integration is discretised using a regular grid of N points $\{x_r\}$, where $x_r = r\Delta x$ and Δx is small compared with variation in the surface and in the field ψ_{inc} incident upon it. The integrals above are thus written as sums over subintegrals, over which the slowly-varying terms are treated as constant, and the first is written

$$\hat{\psi}_{inc} \equiv \hat{G}\hat{\psi} \tag{12}$$

where $\hat{\psi}$ denotes the vector $\psi_m \equiv \psi(x_m, h(x_m))$ and G is the matrix

$$\hat{G}_{n,r}(x,z) = \delta_m + \int_{x_{r-1}}^{x_r} G(x,z;x',z')dx' \tag{13}$$

The integrals may be treated semi-analytically, and the above matrix equation is inverted to solve for the field at the surface. The second integral is discretised in the same way to yield the field at any point in the medium. This possibility is considered below.

10 INVERSE PROBLEM - RECONSTRUCTION OF ROUGH SURFACES FROM SCATTERED DATA

A major problem in the study of surface scattering is that of recapturing the surface explicitly from scattered data. There are numerous applications, from radar to underwater acoustics. Despite this motivation, there has been little progress for complicated surfaces, and in practice treatment is largely limited to iterative methods, alough direct methods have appeared recently (eg [7]).

The parabolic equation regime provides a highly efficient method of solving this problem [8], as seen in the example below.

Data:

Consider an irregular surface illuminated by a Gaussian beam incident at a low grazing angle. Suppose that data for the scattered field ψ_s is available at two horizontal planes, say z_1, z_2.

Denote the two functions $\psi_s(x,z_j)$ by ψ_j, where j = 1, 2, ie.

$$\psi_j = \psi_s(x,z_j) \tag{14}$$

Solution:

The above data allow the inverse problem to be formulated as a pair of *coupled integral equations*, in the following way:

Provided z is large compared with the surface variation, the exponent in the Green's function G can be approximated as

$$\frac{ik[z-h(x')]^2}{2(x-x')} \cong ik\frac{z^2-2zh(x')}{2(x-x')} \tag{15}$$

(The error is the factor $exp[ikh^2/2(x-x')]$; although this exponent becomes large as x' approaches x, the phase variation in G is nevertheless dominated by the approximate exponent).

Now define the function

$$E(x)=\exp[-ikh(x)] \tag{16}$$

Recall that the scattered field at depth z is related to the surface by the integral

$$\psi_s(\mathbf{r})=\int_0^\infty G(\mathbf{r};\mathbf{r}')\frac{\partial\psi(\mathbf{r}')}{\partial z}dx' \tag{17}$$

where $\mathbf{r}'$ is a point on the surface and $\mathbf{r}$ = (x,z) is any point in the medium.

Using the above expressions we can now relate the scattered data to the surface by

$$\psi_s(x,z_j)\cong\alpha\int_0^x \frac{\exp\left[\frac{ikz_j^2}{2(x-x')}\right]}{\sqrt{x-x'}}E(x')^{z_j/(x-x')}\frac{\partial\psi(x')}{\partial z}dx' \tag{18}$$

for j = 1, 2, where the surface-dependent functions E and $\partial\psi/\partial z$ appearing here are independent of the depth z_j. this equation for j = 1, 2 thus represents a pair of coupled integral equations, in which the left-hand sides are known, and the two unknown functions (including E which forms part of the kernel) appear under the integral sign.

Write these equations as

$$\{\psi_1,\psi_2\}\cong A\{E,\partial\psi/\partial z\} \tag{19}$$

where ψ_j represents the data (ie the vector $\psi_s(x,z_j)$). The problem is then to invert the non-linear operator A.

Numerical Implementation:

In the treatment of this coupled system, the parabolic or 'one-way' nature of the equations is crucial. In effect, the inversion of the operator A is achieved by a generalization of Gaussian elimination. the form of the surface follows immediately from knowledge of exp(ikh), or for moderate surface variation by a simple transformation of $\partial\psi / \partial z$.

Making the simple assumption that both $\partial\psi / \partial z$ and $E^{z_j/(x-x')}$ vary slowly over each subinterval compared with the deterministic variation, these functions can be taken outside the integrals and for j = 1, 2, the equations may be written

$$\psi_s(x_n, z_j) \equiv \sum_{r=1}^{n} E_r^{z_j/(x_n - X_r)} \psi'(x_r)\beta_{n,r}(z_j) \tag{20}$$

where X_r is the mid-point $(x_r + x_{r+1})/2$, $E_r = E(x_r)$

$$\beta_{n,r}(z_j) \equiv \alpha \int_{x_r}^{x_{r+1}} \frac{\exp\left[\frac{ikz_j^2}{2(x-x')}\right]}{\sqrt{x-x'}} dx' \tag{21}$$

for j = 1, 2. The coefficients β, which depend only on n-r, may be found exactly in terms of Fresnel integrals. We then obtain a pair of matrix equations, with lower-triangular matrices $\beta(z_j)$, which may be solved simultaneously from the left.

11 NON-PARABOLIC GENERALIZATION FOR DIRECTIONAL SCATTERING

The parabolic equation method has major advantages, but also distinct limitations. The main advantages are high computational efficiency, and that it is analytically tractable; both these result from its uni-directional form. However, the method is valid only for small slopes because of the requirements of small angles of scatter, and backscatter is ignored; in addition the analytical treatment described above is valid in the perturbation region of small surface roughness. This raises the question: can the technique be generalized to include wide-angle scattering and rougher surfaces?

DIRECTIONAL OPERATOR SPLITTING

The advantages of the parabolic equation method can be generalized to the full wave equation in the following way: Consider the Helmholtz integral equation (for given boundary conditions) which relates the incident field to the unknown field values at the surface. This can be written in operator form as

$$\psi_{inc}(\mathbf{r}) = \mathbf{A}\phi \tag{22}$$

where (in the two-dimensional case) $\mathbf{r} = (x, h(x))$, ϕ denotes the field function sought ast the surface, and $\mathbf{A}$ is an integral operator depending upon the rough surface. we can in general split the range of integration at each horizontal point x. This gives rise to a pair of integral operators, the 'left' and 'right' components of $\mathbf{A}$, and we obtain

$$\psi_{inc}(\mathbf{r}) = (\mathbf{L} + \mathbf{R})\phi \tag{23}$$

Series Solution:

Now suppose that the field is incident from the left. Then, although the operators **L** and **R** are comparable in norm, their action on ϕ is highly dependent on the angle of incidence, and $\mathbf{L}\phi$ may be substantially larger that $\mathbf{R}\phi$. We can thus expand the inverse of this equation in a series, and write

$$\phi = (1 - \mathbf{RL}^{-1} + (\mathbf{RL}^{-1})^2 - \ldots)\mathbf{L}^{-1}\psi_{inc} \qquad (24)$$

This expression can be truncated, say at the second term, or treated simply as an iterative series in directional terms. Since **L** represents scattering from the left, when discretized it gives rise to a **lower-triangular matrix**, so numerical inversion of the corresponding matrix equation is extremely efficient. In addition, each term in the series contains multiple scattering effects, and the leading order backscatter is exhibited by the first two terms. Further analytical work is needed to evaluate statistics from this series, and it is expected that this will shed considerable light on the interaction of the left and right-going components of the scattered wave.

12 REFERENCES

[1] E. Thomas, Rough surface scattering using the parabolic equation, J. Acoust. Soc. Am., Suppl 1 **82** 5103 (1987)

[2] M. Spivack, A numerical approach to rough surface scattering by the parabolic equation method, J. Acoust. Soc. Am. **87** 1999-2004 (1990)

[3] J.G. Watson and J.B. Keller, Rough surface scattering via the smoothing method, J. Acoust. Soc. Am. **75** 1705-1708 (1984)

[4] N. C. Bruce, V. Ruizcortes and J. C. Dainty, Calculation of grazing incidence scattering from random rough surfaces using the Kirchhoff approximation, Opt. Comm. **104** 123-126 (1994)

[5] M. Spivack, Coherent field and specular reflection at grazing incidence on a rough surface, J. Acoust. Soc. Am. **95** 694-700 (1994)

[6] M. Spivack, Moments and angular distribution of intensity for rough surface scattering, J. Acoust. Soc. Am. to appear (1995).

[7] R. S. Wombell and J. A. De Santo, Reconstruction of rough surface profiles with Kirchhoff approximation, J. Opt. Soc. A **8** 1892-1897 (1991)

[8] M. Spivack, Direct solution of the inverse problem for rough surface scattering at grazing incidence, J. Phys. A: Math. and Gen. **25** 3295-3302 (1992)