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DYNAMICS OF A LONGITUDINALLY VIBRATING ROD WITH VARIABLE LENGTH AND CROSS-SECTION

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A linear model of longitudinal vibration is formulated for a viscoelastic rod subjected to external harmonic excitation within the framework of the classical theory of vibrating rods. It is assumed that the rod has a time-dependent variable length and cross-section. A mixed problem of dynamics is formulated, which contains non-conventional fixed-free boundary conditions with the coordinate on the right-hand side of the rod being dependent on time. A special transformation of variables eliminates the dependence of the right-hand side coordinate of the boundary conditions on time. The transformation substantially simplifies the boundary conditions, converting them to the classical fixed-free boundary conditions. The simplification of the boundary conditions is, in turn, exacerbated by the equation of rod motion because it becomes a linear partial differential equation with variable coefficients containing some additional terms. The proposed solution of this equation is built in terms of a trigonometric series with time-dependent coefficients, where the spatial components satisfy the boundary conditions. In this case the original partial differential equation is converted into an infinite system of coupled ordinary differential equations with corresponding initial conditions. Truncation of the system produces an initial problem which is solved numerically. The corresponding truncated trigonometric series rapidly converges to the solution. The numerical solution is built for forced oscillations of a lightly damped rod with a simultaneously increasing length and proportionally decreasing area of cross-section so that the volume of the rod remains constant, i.e.in the case of equivoluminally growing rod.
Keywords: longitudinal vibration, rod, variable length, variable cross-section, resonance

1. Introduction

The development of mathematical theories describing the processes of modern additive manufacturing technologies is a subsection of the topical problems of theoretical and applied mechanics [1, 2]. New mathematical models describing the strain-stress evolutions in the process of additive manufacturing were developed in [3]. The mathematical models of continuously growing bodies in the case of small and finite deformations were presented in [8]. Non-stationary dynamical problems of growing bodies are now of special interest. A transient dynamical problem for an accreted thermoelastic parallelepiped was considered in [9, 10]. The present paper is devoted to the longitudinal vibration of a slender rod with varying geometrical parameters such as length and cross-section. Previously analogous problems were considered in the works of some of the co-authors of the present paper [16-18]. A Kelvin viscoelastic rod of initially unit length, which is actuated by a harmonic force, is considered in the framework of the classical theory of rods. It is assumed that the left end of the rod is fixed and the free right end moves with small linear velocity so that the length of the rod increases linearly. Moreover, it is assumed that the cross-section area of the rod simultaneously increases or decreases in accordance with the power law. The corresponding mixed (boundary-initial) problem is formulated mathematically so that the coordinate of the right-hand side boundary condition depends on time. A transformation of coordinates is proposed, which converts the given unconventional boundary-value problem into the standard one. This transformation simplifies the boundary conditions but simultaneously substantially complicates the governing partial differential equation coefficients which become time dependent. Assuming the damping effects are small, we neglect the effects of second-order smallness as well as the second-order effects of small longitudinal growth rate. The simplified equation obtained is approximately solved by its transformation into a truncated system of ordinary differential equations with corresponding initial conditions. The assumed truncated series form of the solution is formulated so that the boundary conditions are automatically satisfied. Numerical simulation of the vibrating rod with variable geometric dimensions is for an equivoluminally growing rod with light damping.

2. Equation of motion and its transformation

In this section we describe the mathematical model of a longitudinally vibrating cylindrical medium's geometrical parameters, which are changed in both the longitudinal and/or lateral directions. This model can be used to describe either an elastic solid rod or an ideal compressible fluid (acoustic medium) enveloped in a rigid cylindrical shell with variable geometry and a soft membrane on the right-hand side. In all of these cases we will refer to the deformed and vibrating medium as a "rod". The equation of motion of a longitudinally vibrating rod actuated by force $f_0 \sin(\omega t)$ in the *Ox*-direction is

$$\frac{\partial}{\partial t}\left(A(t)\frac{\partial u}{\partial t}\right) - 2dA(t)\frac{\partial^3 u}{\partial t\,\partial x^2} - c^2 A(t)\frac{\partial^2 u}{\partial x^2} = f_0 \sin(\omega t) \tag{1}$$

where $u = u(t,x)$ is the longitudinal (particle) displacement of the rod and $A(t) = A_0(1+\varepsilon\eta t)^\alpha$ is its time-dependent cross-sectional area . It is assumed that the change of the length of the rod is proportional to time: $l(t) = l_0(1+\varepsilon t)$. Let us assume that $l_0 = 1$, $\eta = const$, α is positive, negative or zero constant and ε is positive or negative "small" constant ($0 < |\varepsilon| << 1$). The physical meaning of ε is the rate of growth (if $\varepsilon > 0$) or shrinkage (if $\varepsilon < 0$) of the rod's length. The dissipative factor d which is also assumed to be small is proportional to the viscoelastic damping in the Kelvin-Voight model. The term $2dA(t)\frac{\partial^3 u}{\partial t\,\partial x^2}$ describes wave dissipation in the viscous heat transfer medium. In this case $2d = \frac{\rho}{A_0}\left[\frac{4\eta}{3} + \zeta + \kappa\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right]$, where η, ζ are correspondingly volumetric and shear viscosities, κ is thermal conductivity, and c_v, c_p are heat capacities of the medium at constant volume and pressure, respectively. The phase velocity of the longitudinal wave propagation is $c = \sqrt{\frac{E}{\rho}}$, where E is the modulus of elasticity and ρ is the mass density of the rod. The amplitude of periodic excitation of the rod, f_0, is proportional to the excitation force, and ω is the (angular) frequency of periodic excitation. It is also possible to consider a compressible fluid stem oscillating in a longitudinal direction in the framework of equation (1) with $c = \sqrt{\frac{K}{\rho}}$, where K is the bulk modulus of the fluid and ρ is its mass density.

Boundary conditions for the rod are as follows:

$$\begin{aligned} x = 0: &\qquad u(t, x=0) = 0 \\ x = 1+\varepsilon t: &\qquad \frac{\partial u}{\partial x}(t, x = 1+\varepsilon t) = 0 \end{aligned} \tag{2}$$

The initial conditions are

$$t=0: \qquad u(t=0,x)=0, \qquad \frac{\partial u}{\partial t}(t=0,x)=0 \tag{3}$$

Let us make a change of variables $(t,x)\rightarrow(\tau,y)$ in the mixed problem (1) – (3) as follows:

$$t=\tau, \qquad x=y(1+\varepsilon\tau) \tag{4}$$

After making the change of variables (4) in equation (1) and after its division by $A(\tau)$, the equation of motion is:

$$\frac{\partial^2 v}{\partial\tau^2}-\frac{2d}{(1+\varepsilon\tau)^2}\frac{\partial^3 v}{\partial\tau\,\partial y^2}-\frac{2\varepsilon y}{1+\varepsilon\tau}\frac{\partial^2 v}{\partial\tau\,\partial y}+\frac{2\varepsilon d y}{(1+\varepsilon\tau)^3}\frac{\partial^3 v}{\partial y^3}+\frac{2\varepsilon\eta\alpha}{1+\varepsilon\eta\tau}\frac{\partial v}{\partial\tau}$$
$$+2\varepsilon^2\left[\frac{1}{(1+\varepsilon\eta\tau)^2}-\frac{\eta\alpha}{(1+\varepsilon\tau)(1+\varepsilon\eta\tau)}\right]y\frac{\partial v}{\partial y} \tag{5}$$
$$-\left[\frac{c^2-\varepsilon^2y^2}{(1+\varepsilon\tau)^2}-\frac{4\varepsilon d}{(1+\varepsilon\tau)^3}\right]\frac{\partial^2 v}{\partial y^2}+\frac{2\varepsilon^2\eta^2\alpha(\alpha-1)}{(1+\varepsilon\eta\tau)^2}v=\frac{f_0}{A_0}\frac{\sin(\omega\tau)}{(1+\varepsilon\eta\tau)^\alpha}$$

where $v=v(\tau,y)=u(t=\tau,x=y(1+\varepsilon\tau))$. The new boundary conditions are

$$\begin{aligned} y=0: &\qquad v(\tau,y=0)=0 \\ y=1: &\qquad \frac{\partial v}{\partial y}(\tau,y=1)=0 \end{aligned} \tag{6}$$

and new initial conditions are

$$\tau=0: \qquad v(\tau=0,y)=0, \qquad \frac{\partial v}{\partial\tau}(\tau=0,y)=0 \tag{7}$$

Equation (5) contains several terms of higher than $O(\varepsilon)$ and $O(d)$ order of smallness. Neglecting terms $O(\varepsilon^2)$ and $O(\varepsilon d)$ in equation (5), the simplified equation is written as follows:

$$\frac{\partial^2 v}{\partial\tau^2}-\frac{2\varepsilon y}{1+\varepsilon\tau}\frac{\partial^2 v}{\partial\tau\,\partial y}-\frac{2d}{(1+\varepsilon\tau)^2}\frac{\partial^3 v}{\partial\tau\,\partial y^2}+\frac{2\varepsilon\eta\alpha}{1+\varepsilon\eta\tau}\frac{\partial v}{\partial\tau}-\frac{c^2}{(1+\varepsilon\tau)^2}\frac{\partial^2 v}{\partial y^2}\approx\frac{f_0}{A_0}\frac{\sin(\omega\tau)}{(1+\varepsilon\eta\tau)^\alpha} \tag{8}$$

The boundary and initial conditions for (8) are the same for equations (6) and (7). Let us now consider methods of solution for the formulated mixed problem given by equations (6) to (8).

3. Theoretical treatment: solution of mixed problem (6) – (8)

Let us assume that the solution of problem (6) to (8) is as follows:

$$v=v(\tau,y)=\sum_{m=1}^{\infty}C_m(\tau)\sin\left[\frac{(2m-1)\pi}{2}y\right], \tag{9}$$

This series obviously satisfies boundary conditions (6). Substituting (9) in (8), multiplying the result by $2\sin\left[\frac{(2n-1)\pi}{2}y\right],(n=1,2,\ldots)$ and integrating with respect to y from 0 to 1, the following infinite system of ordinary differential equations is obtained:

$$\frac{d^2C_m}{d\tau^2}+\left[\frac{d\pi^2(2m-1)^2}{2(1+\varepsilon\tau)^2}+\frac{2\varepsilon\eta\alpha}{1+\varepsilon\eta\tau}-\frac{\varepsilon}{1+\varepsilon\tau}\right]\frac{dC_m}{d\tau}+\frac{c^2\pi^2(2m-1)^2}{4(1+\varepsilon\tau)^2}C_m$$
$$+\frac{\varepsilon}{1+\varepsilon\tau}\sum_{\substack{n=1\\(n\neq m)}}^{\infty}\frac{(-1)^{m+n+1}\cdot(2n-1)^2}{(n-m)(m+n-1)}\frac{dC_n}{d\tau}\approx\frac{4f_0}{(2m-1)\pi A_0(1+\varepsilon\eta\tau)^\alpha}\sin(\omega\tau) \tag{10}$$

where $m=1,2,\ldots$. Note that at $n=n_0=const$: $\lim_{m\to\infty}\left[\frac{(2n-1)^2}{(m-n)(m+n-1)}\right]=0$, and at $m=m_0=const$: $\lim_{n\to\infty}\left[\frac{(2n-1)^2}{(n-m)(m+n-1)}\right]=4$. At low frequency excitation of the rod (the classical theory of longitudinal vibration of rods describes exclusively the low frequency vibrations of rods), the system of equations (10) converges rapidly to a solution, and it is thus possible to consider a truncated system of ordinary differential equations $(m=1,2,\ldots,N)$. The explicit form of system (10) at $N=4$ is as follows:

$$\begin{aligned}
&\frac{d^2C_1}{d\tau^2}+\left[d\frac{\pi^2}{2(1+\varepsilon\tau)^2}+\frac{2\varepsilon\eta\alpha}{1+\varepsilon\eta\tau}-\frac{\varepsilon}{1+\varepsilon\tau}\right]\frac{dC_1}{d\tau}+\frac{c^2\pi^2}{4(1+\varepsilon\tau)^2}C_1\\
&+\frac{\varepsilon}{1+\varepsilon\tau}\left(\frac{9}{2}\frac{dC_2}{d\tau}-\frac{25}{6}\frac{dC_3}{d\tau}+\frac{49}{12}\frac{dC_4}{d\tau}\right)\approx\frac{4f_0}{\pi A_0(1+\varepsilon\eta\tau)^\alpha}\sin(\omega\tau)\\
&\frac{d^2C_2}{d\tau^2}+\left[d\frac{9\pi^2}{2(1+\varepsilon\tau)^2}+\frac{2\varepsilon\eta\alpha}{1+\varepsilon\eta\tau}-\frac{\varepsilon}{1+\varepsilon\tau}\right]\frac{dC_2}{d\tau}+\frac{9c^2\pi^2}{4(1+\varepsilon\tau)^2}C_2\\
&+\frac{\varepsilon}{1+\varepsilon\tau}\left(-\frac{1}{2}\frac{dC_1}{d\tau}+\frac{25}{4}\frac{dC_3}{d\tau}-\frac{49}{10}\frac{dC_4}{d\tau}\right)\approx\frac{4f_0}{3\pi A_0(1+\varepsilon\eta\tau)^\alpha}\sin(\omega\tau)\\
&\frac{d^2C_3}{d\tau^2}+\left[d\frac{25\pi^2}{2(1+\varepsilon\tau)^2}+\frac{2\varepsilon\eta\alpha}{1+\varepsilon\eta\tau}-\frac{\varepsilon}{1+\varepsilon\tau}\right]\frac{dC_3}{d\tau}+\frac{25c^2\pi^2}{4(1+\varepsilon\tau)^2}C_3\\
&+\frac{\varepsilon}{1+\varepsilon\tau}\left(\frac{1}{6}\frac{dC_1}{d\tau}-\frac{9}{4}\frac{dC_2}{d\tau}+\frac{49}{6}\frac{dC_4}{d\tau}\right)\approx\frac{4f_0}{5\pi A_0(1+\varepsilon\eta\tau)^\alpha}\sin(\omega\tau)\\
&\frac{d^2C_4}{d\tau^2}+\left[d\frac{49\pi^2}{2(1+\varepsilon\tau)^2}+\frac{2\varepsilon\eta\alpha}{1+\varepsilon\eta\tau}-\frac{\varepsilon}{1+\varepsilon\tau}\right]\frac{dC_4}{d\tau}+\frac{49c^2\pi^2}{4(1+\varepsilon\tau)^2}C_4\\
&+\frac{\varepsilon}{1+\varepsilon\tau}\left(-\frac{1}{12}\frac{dC_1}{d\tau}+\frac{9}{10}\frac{dC_2}{d\tau}-\frac{25}{6}\frac{dC_3}{d\tau}\right)\approx\frac{4f_0}{7\pi A_0(1+\varepsilon\eta\tau)^\alpha}\sin(\omega\tau)
\end{aligned}\tag{11}$$

4. Numerical Example

As an example we consider forced oscillations of a lightly damped rod with a simultaneously increasing length and proportionally decreasing area of cross-section so that the volume of the rod remains constant (“equivoluminal growth” of the rod). In this case the parameters of the rod are assumed to be equal to $c=1,\ d=0.05,\ \varepsilon=0.05,\ A_0=1,\ \alpha=-1,\ \eta=1,\ f_0=1,\ \omega=\frac{4}{3}$. The solution to the problem at time interval $\tau\in[0,750]$ is shown in Figure 1.

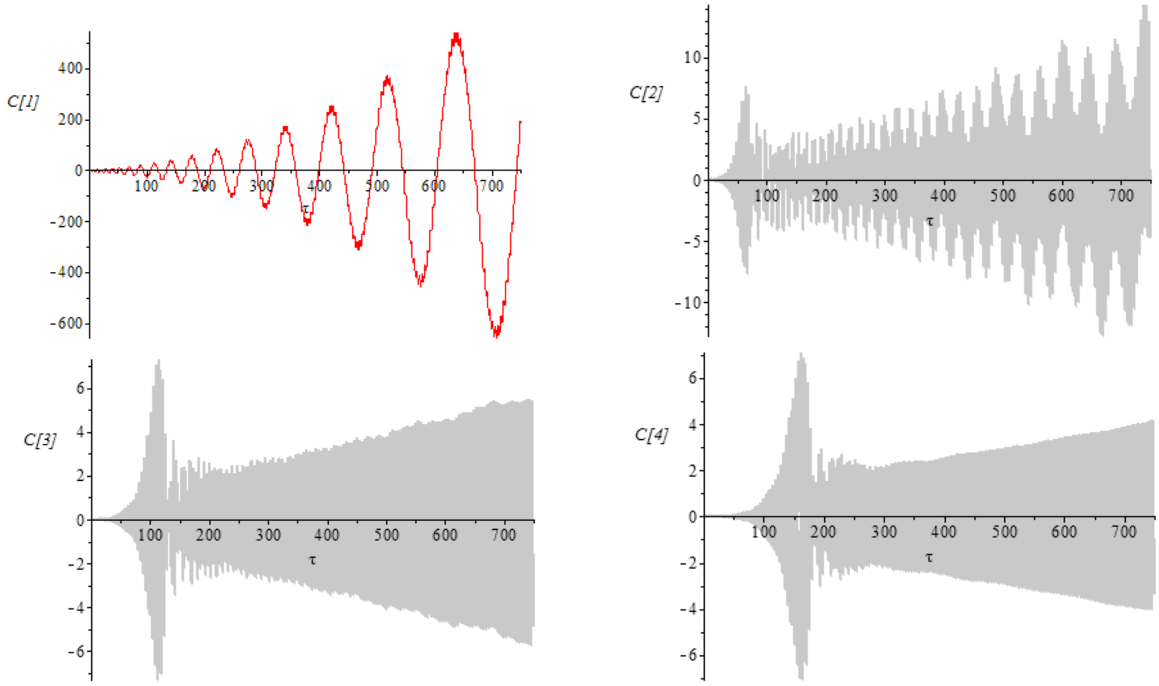

Figure 1: Four modes of forced vibration of "equivoluminally growing" rod with light damping ($c=1,\ d=0.05,\ \varepsilon=0.05,\ A_0=1,\ \alpha=-1,\ \eta=1,\ f_0=1,\ \omega=\frac{4}{3},\ \tau\in[0,750]$)

It follows from this simulation that the first mode starts to increase amplitude of its free oscillations according to a quadratic law. One can observe that this mode is represented as a superposition of quadratically growing natural oscillations and growing forced oscillations at excitation frequency ω. Equations (10) and (11) suggest that the amplitude of the excited vibrations grows linearly due to the linear growth of excitation force at $\alpha=-1$. Higher harmonics also demonstrate vibrations with increasing amplitude, but the rate of amplitude growth is less than that of the first harmonic due to the increase (proportionally to $(2m-1)^2$) of the equivalent damping factor.

5. Conclusions

A linear model of longitudinal vibration is formulated for a viscoelastic rod with variable length and variable cross-section in the classical framework. It is assumed that the rod is subjected to external harmonic excitation. A mixed problem of dynamics is formulated which contains fixed-free non-conventional boundary conditions where the coordinate of the right end of the rod is time dependent. A transformation of variables is proposed, which eliminates the dependence of the right-hand side coordinate of the boundary conditions on time. This transformation substantially simplifies the boundary conditions converting them to the classical ones. The trade-off of the boundary condition simplification leads to a substantial complication of the equation of rod motion, which becomes a linear partial differential equation with variable coefficients. A solution of this equation is sought in terms of trigonometric series with time dependent coefficients, where the spatial components satisfy the boundary conditions. By means of this representation of the solution, the original partial differential equation is converted into an infinite system of ordinary differential equations with corresponding initial conditions which are solved numerically. Truncation of the system generates a truncated trigonometric series which rapidly converges to the solution. A numerical solution is built for a lightly damped rod with a simultaneously increasing length and proportionally decreasing area of cross-section so that the volume of the rod remains constant, i.e. for the so-called equivoluminally growing rod, which demonstrates vibrations with increasing amplitude.

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