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PRESSURE CORRECTION METHOD FOR NON-LINEAR DUCT ACOUSTICS

M M Titahmboh & K S Peat

Department of Mathematical Sciences, Loughborough University, Leics

1. INTRODUCTION

Non-linear problems in duct acoustics are usually analysed by either charactersistic or finite difference/volume methods. The latter have a definite advantage for three-dimensional solutions, for which the methods are a simple extension of the one-dimensional analysis. Finite difference/volume schemes are well-proven for the analysis of highly non-linear wave propagation, including shock-capturing, and are widely used in engine cycle analysis. However, such schemes have not proved so useful for the acoustic analysis of exhaust silencer systems, where it is necessary to accurately characterise the propagation characteristics of the entire exhaust pressure pulse over an engine cycle. Such a pulse is typically composed of an initial large amplitude oscillation followed by a sequence of small amplitude oscillations. The normal explanation for the inaccuracy in the analysis of the radiated sound pressure is uncertainty in the boundary condition at the open end of the system, the tailpipe exit. However, whilst trying to investigate this problem by the use of a finite volume model of the entire duct and radiated sound field, an entirely different problem was brought to light. The finite volume model was first used to evaluate the radiation condition in the limit of small acoustic amplitudes, in order to compare with the analytical results available from linear duct acoustics. It was found that the finite volume analysis did not converge. Subsequent work showed that the problem persisted for simple one-dimensional duct acoustic problems and led to a comparison of numerical schemes for the one-dimensional analysis of small amplitude acoustic waves.

In this paper, it is shown that the conventional finite difference/volume schemes for non-linear wave propagation are inherently unstable when used to analyse the propagation of small amplitude waves. Since, as noted above, a large proportion of an exhaust pulse consists of small amplitude oscillations, this could be of fundamental importance to the evaluation of radiated sound pressure. A pressure-correction finite difference scheme is then given, which gives accurate results for the propagation of waves of either small or large amplitude.

2. GOVERNING EQUATIONS

The system of equations under consideration is the one dimensional system of Euler equations for inviscid compressible flow:

$$\frac{\partial}{\partial t}\rho+\frac{\partial}{\partial x}(\rho u)=0 \tag{1}$$

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}\left(\rho u^2+p\right)=0 \tag{2}$$

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$$\frac{\partial}{\partial t}(\rho e)+\frac{\partial}{\partial x}(\rho u h)=0 \tag{3}$$

where t is the time, x is the space variable, ρ is the density, u is the velocity, e is the total energy, $p=\frac{\gamma-1}{\gamma}\rho\left[h-\frac{u^2}{2}\right]$ is the pressure and $h=e+\frac{p}{\rho}$ is the total enthalpy.

The application considered in this paper is that of sound propagation in a closed, rigid duct with an oscillating piston at one end and a reflecting boundary at the other end. The boundary conditions used are that fluid at the first boundary, x = 0, has the same velocity as the harmonically driven piston, $u = u_p \cos \omega t$, and that at the other boundary, x = L, the fluid velocity is zero. All other properties required at the boundaries are extrapolated from within the fluid. Initial conditions are for unperturbed fluid everywhere. The examples shown correspond to a driven frequency of 100 Hz which is non-coincident with a resonant frequency of the tube which has length 0.5m, for a speed of sound of 340 m/s.

3. CONVENTIONAL FINITE DIFFERENCE/VOLUME FORMULATIONS

3.1 Finite Volume Formulation

The set of equations (1) to (3) are numerically modelled using a Cell-Centered Finite Volume scheme [1]. In vector form, these equations can be written as

$$\frac{\partial}{\partial t}\underline{v}+\frac{\partial}{\partial x}\underline{f}(\underline{v})=\underline{0}\ . \tag{4}$$

Integrating (4) over a volume $V_i = A.\Delta x_i$, for constant area A, between $x_{i-1/2}$ and $x_{i+1/2}$ gives

$$\left[\frac{\partial}{\partial t}\underline{v}\right]_i=\frac{-1}{\Delta x_i}\left[\underline{f}(\underline{v})\Big|_{x_{i+1/2}}-\underline{f}(\underline{v})\Big|_{x_{i-1/2}}\right], \tag{5}$$

where $\partial \underline{v}/\partial t$ is the average value in the volume V_i. In the computational domain the vector $\underline{v}$ is stored at the center of each volume and the values of vector $\underline{f}$ are needed at the vertical edges of the rectangles. At edge $i+1/2$, for example, $\underline{f}_{i+1/2}=.5\left(\underline{f}(\underline{v}_i)+\underline{f}(\underline{v}_{i+1})\right)$. By substituting this in equation (7) and simplifying, it can be seen that in this simple one-dimensional case the Cell-Centered Finite Volume scheme corresponds to a second order central difference scheme.

3.2 MacCormack's Method

The basis of this scheme [2] is an explicit predictor-corrector approach to the solution of equations (1) to (3). The predictor uses first-order forward differences in both space and time, whilst the corrector is again first-order, but uses forward differences in time and backward differences in space. The entire scheme is second-order. The method captures imbedded shocks but 'smears' them over several grid points. The numerical artificial viscosity needed for smooth and accurate simulation of shock waves in the flow is applied impicitly.

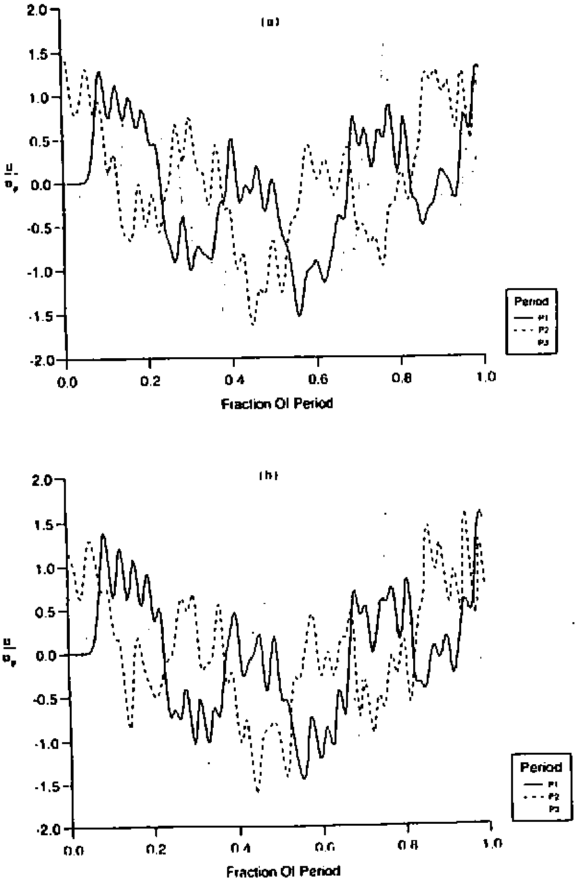

Figure 1. Finite Volume Method. Mid-point velocity variation over the first three cycles for:
(a) u_p = 0.01 m/s and (b) u_p = 10.0 m/s

3.3 Total Variation Diminishing (TVD) Scheme

The particular scheme used was that of Harten [3]. The scheme is again second-order and explicit, but may be considered as an order of magnitude greater in complexity than MacCormack's method. In particular, the scheme advances locally along the characterisic and particle paths and

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thereby retains the advantages of the method of characteristics. The scheme is shock-capturing but, in order to reduce shock 'smearing', the total variation diminishing property is introduced. This and the previous method have been used successfully in the context of engine exhaust gas flows, although they are ideally suited to the determination of steady-state solutions.

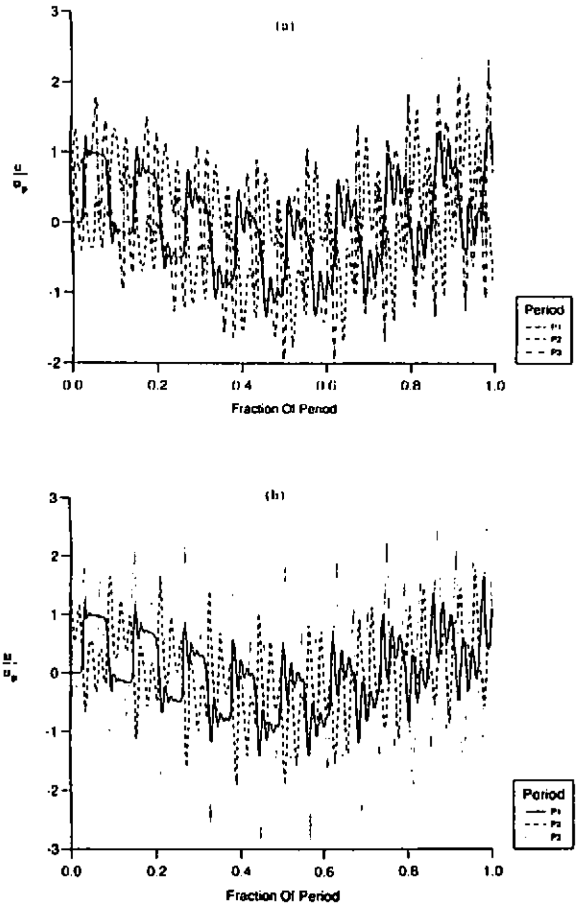

Figure 2. MacCormack's Method. Mid-point velocity variation over the first three cycles for: (a) $u_p = 0.01$ m/s and (b) $u_p = 10.0$ m/s

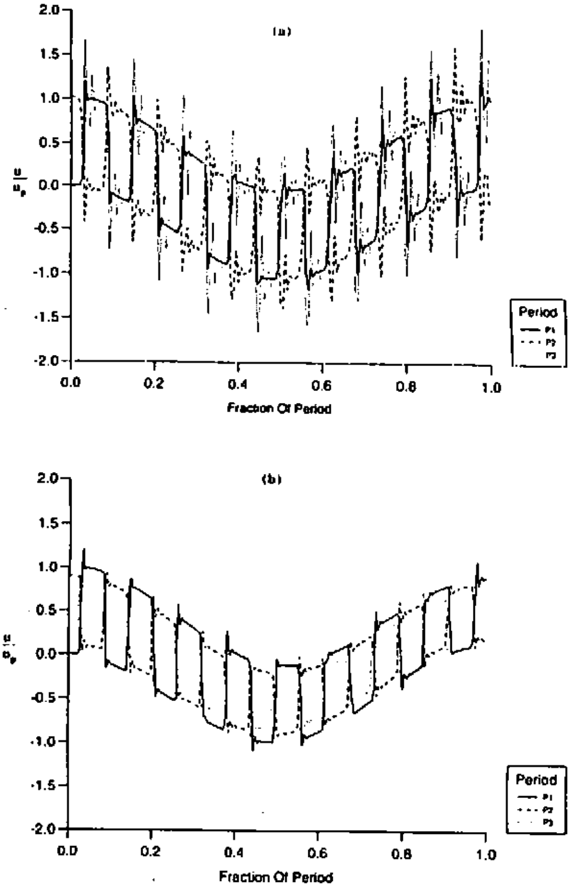

Figure 3. TVD Finite Difference Method. Mid-point velocity variation over the first three cycles for: (a) $u_p = 0.01$ m/s and (b) $u_p = 10.0$ m/s

4. PRESSURE-CORRECTION METHOD

The basis of most pressure-correction methods is that of the Semi-Implicit Method for Pressure Linked Equations (SIMPLE) scheme of Patanker and Spalding [4], applicable to incompressible flows. Further developments gave schemes applicable to compressible flows, and then McGuirk and Page [5] developed a scheme for the solution of flow fields with a wide range of Mach number, from regions of supersonic flow to stagnation regions. Since the time-dependent flow in an exhaust system varies within such wide limits, it seemed appropriate to develop a variant of that scheme for this particular application.

The basic procedure is to advance the momentum field one time step and then to evaluate the pressure corrections required to satisfy the remaining conservation equations on the basis of these advanced momentum values. The pressure corrections are then used to update the density, momentum and energy fields. From the finite difference approximation for equation (2),

$$\frac{1}{\Delta t}\left\{(\rho u)_{i+1/2}-(\rho u)^{0}_{i+1/2}\right\}+\frac{1}{\Delta x}\left\{\bar{u}_{i+1}(\rho u)_{i+1/2}-\bar{u}_i(\rho u)_{i-1/2}+p_{i+1}-p_i\right\}=0, \qquad (6)$$

where the superscript o denotes the value of the property at the beginning of the time step and the overbar denotes values averaged over a cell. Thus

$$A(\rho u)_{i+1/2}+B(\rho u)_{i-1/2}=D\ , \qquad (7)$$

where $A=\bar{u}_{i+1}+\frac{\Delta x}{\Delta t}$, $B=-\bar{u}_i$ and $D=\frac{\Delta x}{\Delta t}(\rho u)^{0}_{i+1/2}-\left[p_{i+1}-p_i\right]$.

With given boundary conditions, equation (7) can be solved implicitly by a tridiagonal routine for the updated momentum values.
The next step is to derive the pressure-corrections from this change in momentum. The change equation relating to equation (7), making use of the standard SIMPLE approximation that coefficient A dominates for small enough Δt, is

$$A\,\delta(\rho u_{i+1/2})\approx-\left[\delta p_{i+1}-\delta p_i\right]. \qquad (8)$$

From the continuity equation (1) we have the corresponding finite difference equation:

$$\frac{1}{\Delta t}\left\{\rho_i-\rho_i^0\right\}+\frac{1}{\Delta x}\left\{(\rho u)_{i+1/2}-(\rho u)_{i-1/2}\right\}=\Delta q\neq 0 \qquad (9)$$

where the inequality results from the provisional values of density and momentum. The pressure field is corrected in just the right manner such that the corresponding changes in the density and momentum fields enforce satisfaction of the continuity equation, that is

$$\delta\rho_i+\frac{\Delta t}{\Delta x}\delta\left\{(\rho u)_{i+1/2}-(\rho u)_{i-1/2}\right\}\approx-\Delta q \qquad (10)$$

where the changes in pressure and density are related through the equation of state, namely

$$\delta\rho_i=\frac{\delta p_i}{\left\{\frac{\gamma-1}{\gamma}\left[h_i-\frac{1}{2}u_i^2\right]\right\}}=S_{p\rho i}\delta p_i \qquad (11)$$

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An expression for the pressure corrections now follows from equations (8) to (11) and is of the form

$$a\delta p_i + b\delta p_{i+1} + c\delta p_{i-1} = -\Delta q \qquad (12)$$

for known coefficients a,b,c and Δq.

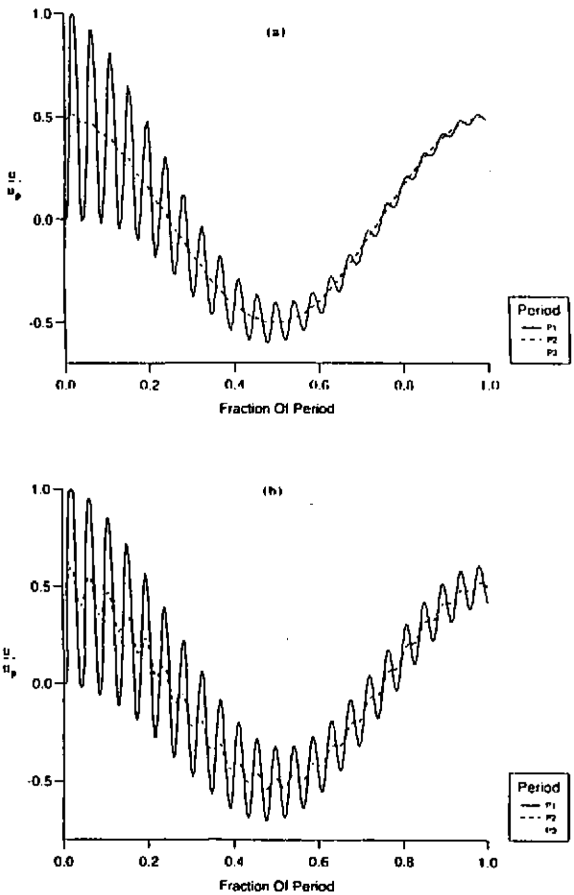

Figure 4. Pressure-Correction Method. Mid-point velocity variation over the first three cycles for: (a) u_p = 0.01 m/s and (b) u_p = 10.0 m/s

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The distribution of δp is calculated using an implicit tridiagonal iteration routine. This done, the pressure distribution and conservation system, via the equations (9,10 and 11), are updated. The energy equation is expressed in the difference form

$$\frac{1}{\Delta t}\left\{(\rho e)_i - (\rho e)_i^0\right\} + \frac{1}{\Delta x}\left\{(\rho u)_{i+1/2} h_i^0 - (\rho u)_{i-1/2} h_{i-1}^0\right\} = 0 \qquad (13)$$

to enable advance of the energy and enthalpy values.

5. DISCUSSION OF RESULTS

Results are presented in Figures 1 to 4 in the form of dimensionless acoustic velocity variations over the first three cycles, for each of the four methods of analysis in turn. The velocity is non-dimensionalised with respect to the piston velocity and the variations of velocity are plotted for the centre point of the duct. Results are given for maximum piston velocities of 0.01 m/s and 10.0 m/s. It is seen that at the lower piston velocity the first three methods give wildly oscillatory solutions which fail to converge. In contrast, the pressure-correction method converges quickly to the quasi steady-state solution. At the higher piston velocity, the first two methods are still non-convergent, while the TVD scheme converges very slowly. The use of artificial viscosity to quell unwanted oscillations works in steady flows, because the damping terms tend to zero as the solution converges, leading, in an asymptotic limit, to the solution of the system. In this case, however, the flow is only quasi-steady so the damping terms will continue to be significant as the solution progresses. The key benefit of the pressure-correction method for small amplitude disturbances would appear to be that errors in the momentum advance are corrected before being fed directly to the continuity equation, which otherwise forces unwarranted large variations in the density and pressure fluctuations.

6. CONCLUSIONS

Many finite difference/volume schemes which are stable when used for high amplitude acoustic pulsations become unstable as the amplitude is reduced. A pressure-correction scheme has been found to be stable at high and low amplitudes and would therefore seem ideal for use in engine exhaust flow analysis, both for overall engine cycle work as well as for the evaluation of radiated noise.

7. REFERENCES

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