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WAVE PROPAGATION IN LAYERED MEDIA

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1. INTRODUCTION

The main objective of this study was to investigate wave propagation in multi-layered highly dissipative materials which are typically used in road construction. The time domain approach is favourable in that obtaining resonance or frequency response data is difficult, if not impossible, due to the requirement of low dissipation at the higher frequencies and wavenumbers which are necessary to produce the required response. Propagation in layered media has historically concentrated upon propagation in soils [1 – 4] and for such considerations the dispersion curves (the relationship between the velocity and wavelength or wavenumber) for typical configurations have been obtained [3,5,6]. The effect of dissipation on these curves has been investigated.

Various investigators have attempted to use the dispersion curves measured on the surface to evaluate the elastic moduli and thicknesses of the various layers. Typically these curves are primarily obtained for the Rayleigh waves (which have vertical and horizontal displacements with amplitudes which decay exponentially with depth from the upper surface) which are dominant in the far field [2,4].

In the near field, where short time histories are typically obtained for impulsive excitation, the body waves (compressive and shear) are important as they contribute significantly and can potentially propagate and reflect from layer interfaces. To obtain the time domain solutions Fourier transform techniques have been applied [4,7] and this has been further extended to dissipative materials, to consider the high time resolution for small layers and for the application of angled impulsive forces. The latter aspect is part of an initial study of specifying a test which essentially can interrogate primarily the upper (top) surface.

2. ANALYTICAL MODELLING OF ELASTIC WAVES IN LAYERED ELASTIC MEDIA

For simplicity the analytical modelling has been restricted to the two dimensional problem where motion in an elastic media is restricted to the direction parallel to the surface (x-direction) and perpendicular to the surface through the depth of the media (z-direction) (see Figure 1). The analysis can be extended to the three dimensional problem using cylindrical coordinates (r,θ,z) (see Brekhovskikh [8]).

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The equations of motion for the displacements u, w in the x- and z-direction respectively are

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu)\frac{\partial \Delta}{\partial x} + \mu \nabla^2 u \tag{2.1}$$

$$\rho \frac{\partial^2 w}{\partial t^2} = (\lambda + \mu)\frac{\partial \Delta}{\partial x} + \mu \nabla^2 w \tag{2.2}$$

where ρ = density of the media, λ,μ = Lamé's constants,

$\Delta = \text{div}\, V \;\; (V = \{u,w\}),$

$= \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z},$ $\qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$

To solve for u and w it is convenient to introduce ϕ, the potential for compressional waves, and ψ, the potential for shear waves, as follows.

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} \tag{2.3}$$

$$w = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \tag{2.4}$$

The potentials are then solutions to the following equations with the boundary conditions on the surfaces of each individual layer.

$$\nabla^2 \phi = \frac{1}{c_1^2}\frac{\partial^2 \phi}{\partial t^2}, \qquad c_1^2 = \frac{\lambda + 2\mu}{\rho} = \text{compression wave speed} \tag{2.5}$$

$$\nabla^2 \psi = \frac{1}{c_2^2}\frac{\partial^2 \psi}{\partial t^2}, \qquad c_2^2 = \frac{\mu}{\rho} = \text{shear wave speed} \tag{2.6}$$

To obtain the dispersion relationship for single, multi-layered and half space problems, it is normal to assume harmonic solutions for ϕ and ψ, i.e.

$$\phi = \phi(z)\, e^{j(\omega t - kx)} \tag{2.7}$$

$$\psi = \psi(z)\, e^{j(\omega t - kx)} \tag{2.8}$$

Substituting equations (2.7) and (2.8) into (2.5) and (2.6) one can obtain solutions for $\phi(z)$, $\psi(z)$ for each layer, i.e.

$$\phi(z) = \frac{A}{\mu}\cos \alpha z - \frac{C}{\mu}\sinh \alpha z \tag{2.9}$$

$$\psi(z) = \frac{iB}{\mu}\cosh \beta z - \frac{iD}{\mu}\sinh \beta z \tag{2.10}$$

where $\alpha^2 = k^2 - k_1^2,\ k_1 = \omega/c_1,\quad \beta^2 = k^2 - k_2^2,\ k_2 = \omega/c_2.$

The four constants A, B, C and D appearing in equations (2.9) and (2.10) are determined from the boundary conditions for the layer. The conditions may be in terms of the displacements u and w (as related to the potentials in equations (2.3) and 2.4)), or in terms of the surface stresses σ_z and τ_{zx}.

$$\sigma_z = \lambda\nabla^2\phi + 2\mu\left(\frac{\partial^2\phi}{\partial z^2}\right) = \lambda\frac{\partial u}{\partial x} + (\lambda + 2\mu)\frac{\partial w}{\partial z} \tag{2.11}$$

$$\tau_{zx} = \mu\left(2\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2}\right) = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{2.12}$$

This leads to a matrix formulation for the equations relating the displacements and stresses with the constants.

$$\begin{Bmatrix} \sigma_z \\ \tau_{zx} \\ w \\ u \end{Bmatrix} = [e_{i,j}] \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} \tag{2.13}$$

$[e_{ij}]$ is a 4×4 matrix whose coefficients are given in reference [9].

For free vibration the non-trivial solution, which allows the dispersion curves to be obtained, is found when the matrix of the equations is singular (i.e. zero determinant). For a half space one has two conditions on the upper surface and conditions that the motion is finite everywhere, i.e. terms in $e^{\alpha z}$, with α positive, must disappear. This is satisfied if A = C and B = D. This criterion leads to only two equations. The other alternative to be considered is a multi-layered media. Each finite depth layer introduces four additional constants, A_i, B_i, C_i, and D_i, but at each interlayer interface there will be four equations due to continuity of the displacements and equilibrium of the stresses, i.e. in general a n-layer system has 4n constants with $4(n-1)$ interface equations and 4 equations from the upper and lower surface. These can be reduced to 4 equations in terms of the upper layer coefficients $A_1 - D_1$ only, using the interface equations and the lower surface equations. This is simplified using matrix algebra [5,6,9].

3. ANALYSIS FOR AN IMPULSIVE FORCE ACTING ON A LAYERED MEDIA

Navier's equations, as given in section 2, are solved by Fourier transformations ([3], [4], [7]) to obtain the time history of the response due to an impulsive force. The models incorporate dissipation via complex elastic moduli and also consider the action of an angled impulse. Consider a normal impulsive force which exerts a uniform pressure P/2d (P is the force/unit length in the y-direction) over a length in the x-direction of 2d. This imposes boundary conditions on the surface as

$$\tau_{zx} = 0 \tag{3.1}$$

$$\sigma_z = \begin{cases} -P/2d & \text{for } |x| < d \\ 0 & \text{for } |x| > d \end{cases} \tag{3.2}$$

The equations of motion for the potentials ϕ and ψ, (2.5) and (2.6), and the boundary conditions (3.1) and (3.2), are transformed using the spatial Fourier transforms as defined below.

$$F(\xi,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x,t)\, e^{-j\xi x}\, dx \tag{3.3}$$

$$f(x,t) = \int_{-\infty}^{+\infty} F(\xi,t)\, e^{+j\xi x}\, d\xi \tag{3.4}$$

Initially one considers the case of a harmonic load and the response to an impulse is given by the standard inverse Fourier transform from the frequency domain ω to the time domain t. Let $\phi = \phi(x,z)\, e^{j\omega t}$ and $\psi = \psi(x,z)\, e^{j\omega t}$, k_1 and k_2 be the compressive and shear wavenumbers respectively and $\bar{\phi}$, $\bar{\psi}$ the Fourier transforms of $\phi(x,z)$ and $\psi(x,z)$ respectively. Then the Fourier transform equivalent of equations (2.5) – (2.6) and (2.9 – (2.10) are

$$\frac{d^2\bar{\phi}}{dz^2} - \alpha^2\bar{\phi} = 0 \tag{3.5}$$

$$\frac{d^2\bar{\psi}}{dz^2} - \beta^2\bar{\psi} = 0 \tag{3.6}$$

where, in this case, $\alpha^2 = \xi^2 - k_1^2$ and $\beta^2 = \xi^2 - k_2^2$.

Solutions for $\bar{\phi}$ and $\bar{\psi}$ are

$$\bar{\phi} = A_1\, e^{-\alpha z} + B_1\, e^{\alpha z} \tag{3.7}$$

$$\bar{\psi} = A_2 e^{-\beta z} + B_2 e^{\beta z} \tag{3.8}$$

Likewise, $\bar{u}$, $\bar{w}$, $\bar{\sigma}_z$ and $\bar{\tau}_{zx}$ can be expressed in terms of the four constants A_1, B_1, A_2 and B_2 and the four exponentials $e^{\pm\alpha z}$ and $e^{\pm\beta z}$ [9].

In particular, for a normal impulse of duration T acting on the surface of a half space the Fourier transformed stresses become

$$\begin{Bmatrix} \bar{\sigma}_z \\ \bar{\tau}_{zx} \end{Bmatrix} = \begin{bmatrix} 2\mu\xi^2 - (\lambda + 2\mu)\, k_1^2 & -j2\mu\beta\xi \\ -j2\mu\alpha\xi & -\mu(2\xi^2 - k_2^2) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} \frac{P}{4\pi}\,\frac{\sin\xi d}{\xi d}\,\frac{\sin\omega T}{\omega} \\ 0 \end{Bmatrix} \tag{3.9}$$

The coefficients B_1, B_2 in this case are both zero (no reflection from $z = \infty$) and hence equation (3.9) in conjunction with the expressions for the transformed potentials (3.7) – (3.8) and the relationship with the displacements (2.3) – (2.4) allow the transformed surface displacements $\bar{u}(\xi,\omega)$ and $\bar{w}(\xi,\omega)$ to be obtained.

$$\bar{u}(\xi,\omega) = D_1 \frac{P}{4\pi}\,\frac{\sin\xi d}{\xi d}\,\frac{\sin\omega t}{\omega} \tag{3.10}$$

$$\bar{w}(\xi,\omega) = D_2 \frac{P}{4\pi}\,\frac{\sin\xi d}{\xi d}\,\frac{\sin\omega t}{\omega} \tag{3.11}$$

where

$$D_1 = \frac{2\mu(\xi - \alpha\beta) - (\lambda + 2\mu)\, k_1^2}{4\mu^3\xi^2\beta(2\xi^2 - k_2^2)\,[2\mu\xi - (\lambda + 2\mu)k_1^2]}$$

and

$$D_2 = \frac{2\mu(\beta^2 - \xi) - (\lambda + 2\mu)\, k_1^2}{4\mu^3\xi^2\beta(2\xi^2 - k_2^2)\,[2\mu\xi - (\lambda + 2\mu)k_1^2]} \tag{3.12}$$

The inverse two dimensional Fourier transform of $\bar{u}$ and $\bar{w}$ was evaluated numerically and the impulse time histories produced. Developments of the approach to a dissipative system of different layers bounded underneath by an elastic half space have been performed. The elastic moduli are taken to be complex, i.e. $E^* = E(1 + j\eta)$. The formulation in terms of four equations (for finite layered space) or two equations (for a media bounded by a half space) leads to the transformed displacements as in equations (3.10) and (3.11) with, in general, more complicated multiplying factors. For an angled impulse ($\pi/2 - \theta$ to the surface normal) the stresses are $\sigma_z = P\sin\theta$ and $\tau_{zx} = P\cos\theta$ and likewise the surface displacements can be evaluated.

4. RESULTS

Figure 1 is an example of the dispersion relationships for three different media showing the branches of the dispersion curves. Indirectly one can observe the difficulty in recognizing a particular thickness/moduli if one does not know in advance the structure of the material. Examples are given in Figures 2 and 3 of the impulse response for a normal and angled impulse respectively. The corresponding frequency response functions for an angled impulse are given in Figure 4 as a function of the thickness of the top layer (on a half space). The computed time and frequency data are for a fairly highly damped structure and indicate the requirement of needing to be near to the source in order to have a reasonable signal to noise ratio. Also, a dominant response in both domains may be interpreted in terms of particular wave types and resonance frequencies from which it may be possible to deduce either material or geometric (thickness) properties.

5. CONCLUSIONS

An analysis for the dispersion curves and the impulse responses for elastic layered media has been developed. In particular, the angled impulse and response may be investigated further to determine at what angle one has a response of the surface which can be directly interpreted for the thickness values. Dissipation and the high frequencies required for interrogation of the upper layer(s) mean that this approach can be used to complement and interpret the results of experimental investigations.

6. REFERENCES

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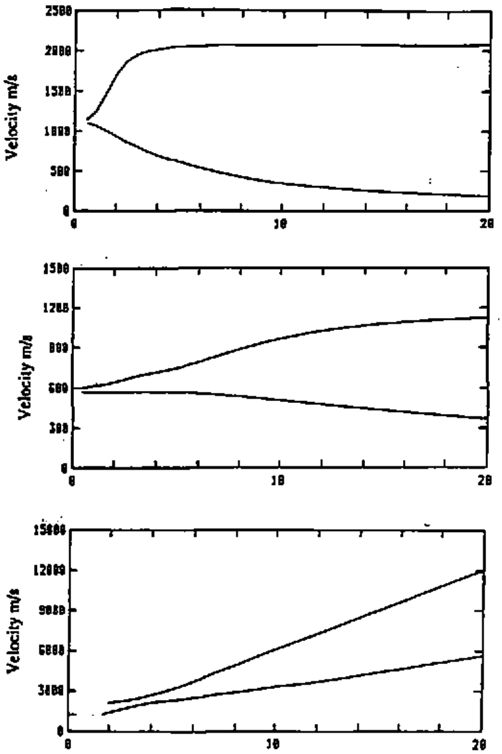
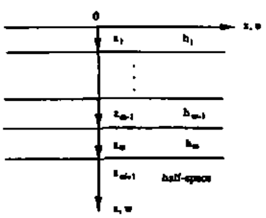


L / H_1 L = wavelength
H_1 = thickness of a top layer.

Figure 1 Velocity dispersions of layered media.
(a) A free-free plate.
(b) Two layered plate.
(c) A layer and half-space.



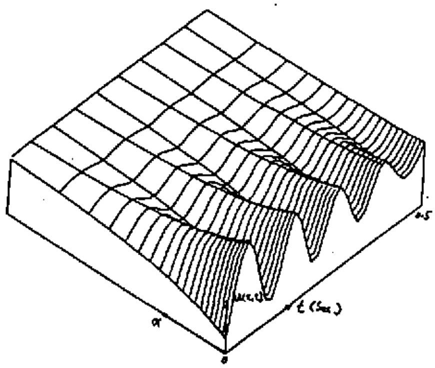
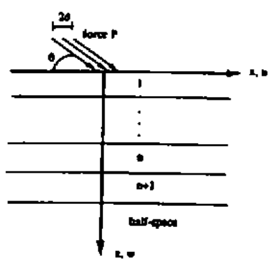


Figure 2. Time History of the near field response (parallel to the surface) due to a normal impulsive force.



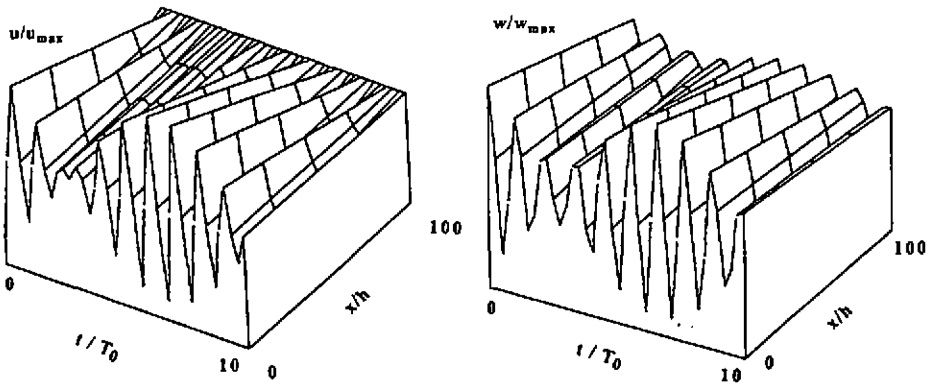
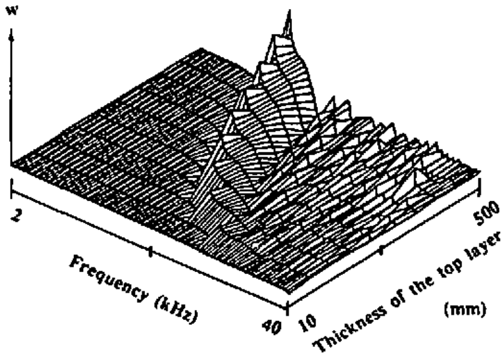


Figure 3. Three dimensional time history for an angled force acting on a layer above a half space.

T_0 = duration of the impulse
t time
x distance from excitation
h thickness of top layer
u displacement parallel to surface (above left)
w displacement normal to surface (above right)



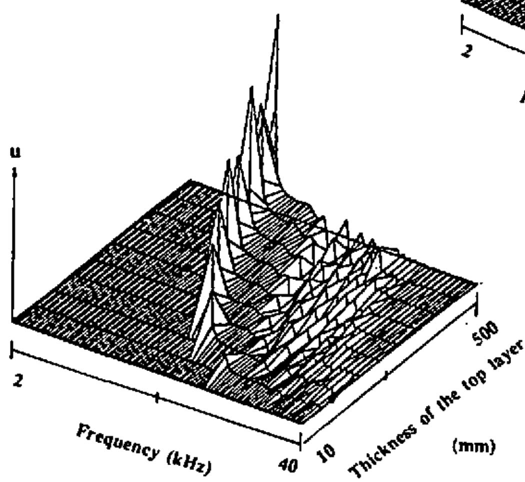


Figure 4. Frequency response for an angled impulse from the time histories in figure 3.