Proceedings of the Institute of Acoustics

INHARMONICITY OF STEPPED STIFF STRINGS.

P Chumnantas, C A Greated & D M Campbell.

University of Edinburgh, Department of Physics, Edinburgh, UK.

1. INTRODUCTION

The problem of the vibration of flexible strings with uniform characteristics has been treated by many investigators and the results are well established. Vibration characteristics of stiff strings are also quite well understood and the predicted mode frequencies are in closed agreement with observations [1]. In this paper, the vibration of nonuniform stiff strings is considered.

In the late 19th century, Lord Rayleigh [2] described a theory for the vibration of strings, showing that in the piano, the stiffness of the strings affects the restoring force to a significant degree. He derived a formula to predict how the stiffness of a piano string can cause it to vibrate at frequencies somewhat greater than those of the ideal string.

The more general theory for the stiff string, often encountered in the literature, was developed by Morse [3], and by Shankland and Coltman [4]. They derived expressions for the frequencies of a string of uniform diameter and density in free transverse vibration between rigid supports. Shankland and Coltman predicted a progressive sharpening of the partials as the mode number increases, the extent of the sharpening being dependent on the ratio of the string diameter to its length; the greater this ratio, the greater will be the sharpening. Robert W. Young [5, 6, 7] and his colleagues found that the sharpening follows approximately a square law with respect to mode number. They observed that the departure from the harmonic series of the plain steel strings was about the same in all the pianos they tested and was consistently less in large pianos than in small ones. More recently, many other investigators have studied the piano string inharmonicity problem with plain steel strings and overwound bass strings [8].

All piano bass strings are characterised by a steel wire core wrapped with copper, or sometimes iron, used to increase the string's linear mass density. While the tight coiling of the copper wire ensures close coupling to the core, the windings contribute considerably more to the increase in the string's linear mass density than to its bending stiffness. Most bass strings have a single winding of copper wire, and it is usually only within the lowest octave that double winding is used. A double-wound string consists of a bare steel core wrapped with a small diameter copper wire, which is then overspun with a second winding of larger diameter. A small part of the steel core is left exposed near the end of the string. Thus only the outer winding is visible and the existence of the inner winding is evident only from the small change in the diameter of the overall covering near the ends.

A theoretical relationship for inharmonicity that can be applied to wrapped strings was derived by Harvey Fletcher [9]. He showed that the formula $f_n = nf\sqrt{1+Bn^2}$ gives values of the partial frequencies of the solid piano strings close to his observed values, where n is the number of the partial. The constant B, the inharmonicity coefficient calculated from the dimensions of the wire, is $B = (\pi^2 QS\kappa^2 / 4l^2 \sigma f_0^2)$, where f_0 is the fundamental frequency, Q the Young's modulus of elasticity, S the area of the cross section, l the length, σ the linear density, and κ the radius of gyration of the string. Fletcher had the idea of applying this to overwound strings by taking σ to be the linear density of the overwound string (core and windings). He suggested that the value of linear density of the overwound string would be

$$\sigma = \rho_{cu}\frac{\pi^2}{16}D^2 + (\rho_s\frac{\pi}{4} - \rho_{cu}\frac{\pi^2}{16})d^2$$

for a steel core of diameter d with volume mass density ρ_s and copper winding with volume mass density ρ_{cu} and wire diameter D.

Fletcher's formula has previously been applied [1] to predict the inharmonicity of strings on a 2.5 m Broadwood grand piano (1871) in the Physics Department at the University of Edinburgh. It was found that for the full range of plain solid strings the predicted and observed inharmonicities were in close agreement. However for the overwound strings the observed inharmonicity was higher than predicted, taking the string as being uniform over its length. The deviation was up to some 30% for the most heavily overwound, A0 string. This has led us to investigate the effect of string nonuniformity, caused by the windings not continuing over the entire string length.

Some discussions about this problem have appeared over the last few years by Levinson [10], Sakata and Sakata [11], and Gottlieb [12]. Levinson studied the free vibration of a string with stepped mass density and derived an exact equation for calculating the natural frequency, but did not obtain any numerical solutions. Sakata and Sakata derived an exact frequency equation for a string with stepped mass density and proposed an approximate formula for estimating the fundamental natural frequency of the string. In Gottlieb's work, the three-part string, with two step discontinuities in density, was investigated in some detail for both fixed and free end conditions. Aspects of the "four-part" and "m-part" string problems were also discussed. However, these derivations have not taken into account the stiffness of the stepped string.

2. THEORETICAL CONSIDERATION

In this section we derive an expression for the frequencies of vibration of a stepped stiff string. Consider the vibration of an M-part string fixed at its ends. The (displacement) finite element formulation of the one-dimensional fourth-order differential equation [2] is

$$T_i\frac{\partial^2 w_i}{\partial x_i^2} - (QS\kappa^2)_i\frac{\partial^4 w_i}{\partial x_i^4} = \rho_i S_i\frac{\partial^2 w_i}{\partial t^2} \qquad i = 1, 2, 3, \ldots, m \qquad (1)$$

INHARMONICITY OF STEPPED STIFF STRINGS.

where w_i is the (small) transverse displacement of the string originally lying on the x-axis, t is the time, T is the Tension, S_i is the area of cross-section, κ_i its radius of gyration. ρ_i and Q_i are the density and modulus of elasticity of the material for $a_{i-1} \le x_i \le a_i$ where x_i is the length of the i-th segment of the string. $a_0 = 0$ and $a_m = \sum_{i=1}^{m} a_i = a$, the total length of the string.

The ends of the string are considered to be clamped. Then the boundary conditions are

$$\begin{aligned} w_1(0) &= w_m(a) = 0 \\ w_1'(0) &= w_m'(a) = 0 \end{aligned} \qquad (2)$$

and the junction conditions

$$\begin{aligned} w_i(a_i) &= w_{i+1}(a_i) \\ w_i'(a_i) &= w_{i+1}'(a_i) \\ (QS\kappa^2)_i w_i''(a_i) &= (QS\kappa^2)_{i+1} w_{i+1}''(a_i) \\ T_i w_i'(a_i) + (QS\kappa^2)_i w_i'''(a_i) &= T_{i+1} w_{i+1}'(a_i) + (QS\kappa^2)_{i+1} w_{i+1}'''(a_i). \end{aligned} \qquad (3)$$

The boundary conditions are those for simple supports and the junction conditions express the continuity of the displacement, slope, moment, and shear at the junctions of the M segments of the stiff string.

In the case of a two segment stiff string, the normal mode frequencies can be found from the equation (afterwards, called the frequency equation):

$$\begin{aligned} &\left(\frac{(QS\kappa^2)_1}{(QS\kappa^2)_2}\frac{\mu_{11}^2}{\mu_{22}^2} + 1\right)(\mu_{11}\tanh(\mu_{21}a_2) + \mu_{21}\tanh(\mu_{11}a_1)) \\ &\quad \times\left\{\frac{(QS\kappa^2)_1}{(QS\kappa^2)_2}\frac{\mu_{12}^2}{\mu_{21}^2} + 1\right\}\left[\mu_{12}\tan(\mu_{22}a_2) + \mu_{22}\tan(\mu_{12}a_1)\right] \\ &\quad\quad -\left\{\frac{(QS\kappa^2)_1}{(QS\kappa^2)_2}\frac{\mu_{11}^2}{\mu_{21}^2} - 1\right\}\{\mu_{11}\tan(\mu_{22}a_2) + \mu_{22}\tanh(\mu_{11}a_1)\} \\ &\quad\quad\quad \times\left\{\frac{(QS\kappa^2)_1}{(QS\kappa^2)_2}\frac{\mu_{12}^2}{\mu_{22}^2} - 1\right\}\{\mu_{12}\tanh(\mu_{21}a_2) + \mu_{21}\tan(\mu_{12}a_1)\} = 0 \end{aligned} \qquad (4)$$

Equation (4) contains four parameters $\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}$ which are functions of the frequency, f.

INHARMONICITY OF STEPPED STIFF STRINGS.

$$\mu_{jk} = \left(\sqrt{\left(\frac{T_j}{2(QS\kappa^2)_j}\right)^2 + (2\pi f)^2 \frac{\rho_j}{(Q\kappa^2)_j}} + (-1)^k \frac{T_j}{2(QS\kappa^2)_j}\right)^{\frac{1}{2}} \qquad (5)$$

$$: j,k = 1,2.$$

In the case of the stepped stiff string it is considered that its tension and stiffness are constant along its length due to the core. Its frequency equation is

$$\left(\frac{\mu_{11}^2}{\mu_{22}^2}+1\right)\left(\frac{\mu_{12}^2}{\mu_{21}^2}+1\right)\left\{\frac{\mu_{11}}{\mu_{21}}\frac{\tanh(\mu_{21}a_2)}{\tanh(\mu_{11}a_1)}+1\right\}\left\{\frac{\mu_{12}}{\mu_{22}}\frac{\tan(\mu_{22}a_2)}{\tan(\mu_{12}a_1)}+1\right\}$$
$$-\left(\frac{\mu_{11}^2}{\mu_{21}^2}-1\right)\left(\frac{\mu_{12}^2}{\mu_{22}^2}-1\right)\left\{\frac{\mu_{11}}{\mu_{22}}\frac{\tan(\mu_{22}a_2)}{\tanh(\mu_{11}a_1)}+1\right\}\left\{\frac{\mu_{12}}{\mu_{21}}\frac{\tanh(\mu_{21}a_2)}{\tan(\mu_{12}a_1)}+1\right\}=0 \qquad (6)$$

The allowed frequencies, f_n : (n = 1, 2, 3, 4, ...) can be found from equation (5) & (6).

3. NUMERICAL RESULT

Numerical calculations have been undertaken to compute theoretical mode frequencies for strings on the Edinburgh Broadwood grand piano. Only the single overwound strings in the lowest octave, sounding A0 to A1 were considered; results here are presented for two of the strings, Bb0 and Db1.

Fig I shows the notation used for defining the parameters of the overwound string. This was clamped at both ends and the linear density was calculated using the method of W T Goddard [13].

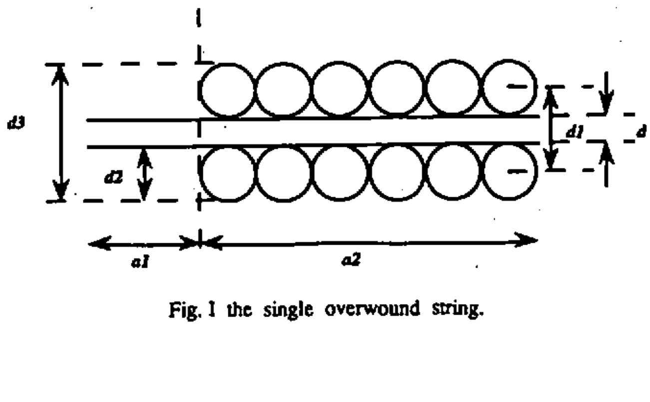


Fig. I the single overwound string.

The 1-st segment is the bare string and the 2-nd segment has both the steel core and the wrapped copper wire. Table I shows the dimensions for the two strings Bb0 and Db1.

Table I The dimensions of the strings. Bb0 and Db1

Strings' parameters	Bb0	Db1
d (mm.)	0.140	0.130
d_2(mm.)	0.441	0.377
a_1(mm.)	23.00	18.00
a_2(mm.)	1837.00	1767.00
σ_1(g/mm.)	0.0121	0.0104
σ_2(g/mm.)	0.1084	0.0794

The mode frequencies were found numerically from Equation 5 & 6 by applying Newton's method; this was programmed on an Apple Macintosh computer using the Mathematica package. The results of these computations are shown in Tables II and III. Figures II and III present these results graphically.

Table II The departure of the natural frequencies from the harmonic series for Bb0 string with Fletcher's formula, Observation and Theory (Eqs. 5 & 6).

Mode number	Bb0		
(n)	Fletcher's formula	Observation	Theory (Eqs. 5 & 6)
1	1.0000	1.0000	1.0000
2	2.0003	2.0005	2.0004
3	3.0011	3.0016	3.0015
4	4.0026	4.0037	4.0035
5	5.0052	5.0072	5.0069
6	6.0089	6.0125	6.0119
7	7.0142	7.0198	7.0190
8	8.0212	8.0296	8.0283
9	9.0301	9.0421	9.0403
10	10.0413	10.0578	10.0552
11	11.0549	11.0769	11.0734
12	12.0713	12.0997	12.0953

Fig II

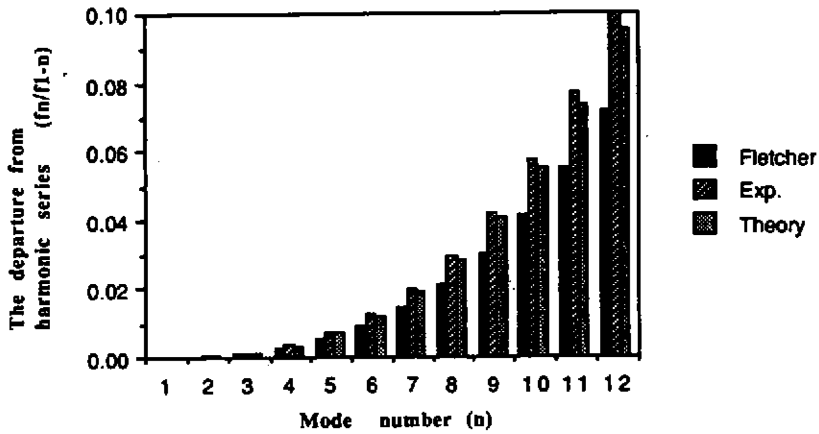
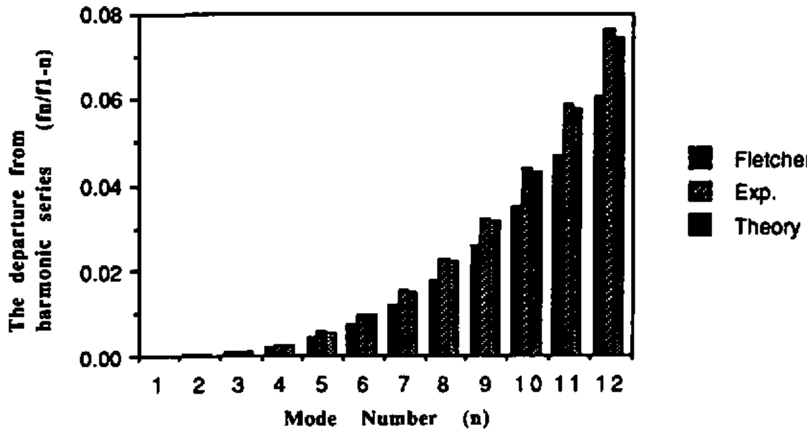


Table III The departure of the natural frequencies from the harmonic series for Db1 string with Fletcher's formula, Observation and Theory (Eqs. 5 & 6).

Mode number (n)	Db1		
	Fletcher's formula	Observation	Theory (Eqs. 5 & 6)
1	1.0000	1.0000	1.0000
2	2.0003	2.0004	2.0004
3	3.0009	3.0012	3.0012
4	4.0022	4.0028	4.0028
5	5.0044	5.0055	5.0054
6	6.0076	6.0095	6.0093
7	7.0120	7.0151	7.0148
8	8.0226	8.0226	8.0221
9	9.0255	9.0321	9.0314
10	10.0350	10.0441	10.0430
11	11.0466	11.0586	11.0573
12	12.0604	12.0761	12.0743

Fig III



4. INHARMONICITY MEASUREMENTS

In order to validate the theory developed in section 2 of this paper, experiments were conducted to measure the inharmonicity of the single overwound strings on the Edinburgh Broadwood. The key of the note under study was held down with a weight in order to retract the damper and allow the string to vibrate freely; on this piano the dampers are below the strings. The string was then plucked with the finger at a position close to the end and the sound was recorded at a point near to the centre of the string using a microphone mounted a short distance above. The acoustic signal was captured digitally using a Barry Box (a unit specially designed for collecting sound samples) and was analysed on a BBC B computer using an FFT routine developed at Edinburgh [1]. This program generates a high resolution spectrum and accurately locates the peaks, from which the inharmonicity of any particular mode can be determined.

The measured peak frequencies for the first twelve modes for string Bb0 are shown in Table II and are displayed graphically in Figure II. These can be compared with the theoretical frequency values, together with the corresponding frequencies calculated from Fletcher's formula. Corresponding results for the note Db1 are given in Table III and Figure III. It is seen that stepped stiff string theory gives a very good match, the error in the twelfth mode being only 5%. The error using Fletcher's theory is approximately six times as great.

Similar results have been measured for the other single overwound strings and the results show generally the same trends.

5. CONCLUSIONS

It is evident from the results that the theory presented here gives a better fit to measured inharmonicities than Fletcher's analysis for a uniform string. Apparently the stepped geometry of the overwound strings is significant. However, our predictions still underestimate the inharmonicity by about 5% in the twelfth mode. This could be due to a number of factors. The winding itself may tend to increase the stiffness of the string i.e. the stiffness of a length of overwound string is slightly greater than the stiffness of the core by itself. The flexibility of the supports may also be important. Neither of these factors are included in the analysis.

In order to study the problem further, a purpose-designed monochord has now been constructed at Edinburgh. With this it will be possible to measure the tension precisely and to vary the support rigidity.

6. REFERENCE

[1] D.M. Campbell, C.A. Greated and R. Parks, *Proc. of the Inst. of Acoustics*, 12(1), 743-752 (1990).
[2] J.W. Strutt Lord Rayleigh, "The Theory of Sound", 2nd ed. Dover Publications, New-York, 1954.
[3] P.M. Morse, "Vibration and Sound", 2nd ed. McGraw-Hill, New York, 1948.
[4] R.S. Shankland and J.W. Coltman, *J. Acoust. Soc. Amer.*, 10(3), 161-1666 (1939).
[5] O.H. Shuck & R.W. Young, *J. Acoust. Soc. Amer.*, 15(1), 1-11 (1943).
[6] R.W. Young, *J. Acoust. Soc. Amer.*, 24(3), 267-273 (1952).
[7] R.W. Young, *Acustica*, 4, 259-262 (1954).
[8] A.J. Bell & R. Parks, *Proc. of the Inst. of Acoustics*, 12(1), 753-755 (1990).
[9] H. Fletcher, *J. Acoust. Soc. Amer.*, 36(1), 203-210 (1964).
[10] M. Levinson, *J. of Sound and Vibration*, 49(2), 287-291 (1976).
[11] T. Sakata & Y. Sakata, *J. of Sound and Vibration*, 71(2), 315-317 (1980).
[12] H.P.W. Gottlieb, *J. of Sound and Vibration*, 118(2), 283-290 (1987).
[13] W. Trow Goddard, "A handbook on the strings of the pianoforte and other keyboard instruments with design formulae", Billingham Press, Billingham, England, 1985.