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STABILITY AND STEADY STATE PERFORMANCE OF CONSTRAINED OUTPUT POWER ADAPTIVE FILTERS

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0. ABSTRACT

This paper describes a deterministic analysis of the behaviour of two classes of adaptive filter with constrained output variance. These filters are useful in the context of the control of distributed parameter systems, such as acoustic spaces, and the control of systems having hard-limiting nonlinearities. Stability limits and steady state performance of the adaptive filters operating in an estimation role are discussed.

1. INTRODUCTION

In practical adaptive sound control applications it is generally desirable to use a minimum power control effort. High output variances from the controlling adaptive filters may exceed the linear power handing envelope of the transducers used as secondary sources. The resulting distortion products will usually be treated as uncorellated additive noise by the adaptive control system. High secondary power can also be symptomatic of a controllability problem, if the secondary source distribution is not efficiently coupled to particular system modes.

The output power can be usefully controlled if it appears explicitly as a component of the "cost function" of the adaptive filter, such that the system attempts to minimise some combination of mean square estimation error and estimation power. This paper describes a deterministic analysis of the performance of two constrained variance adaptive filters. Both of the filters will be familiar to readers interested in adaptive signal processing in general and adaptive control of acoustic systems in particular.

2. MINIMUM OUTPUT VARIANCE MEAN SQUARE ESTIMATION

The simplest constrained variance cost function would be a weighted sum of mean square estimation error and estimation power. A "stochastic gradient" [1] search for the filter configuration representing the minimum of such a function would adjust the filter's weights, W, according to the instantaneous gradient of the performance surface. The cost function may be written as:

$$J_T = e^2 + alpha_1 * y^2 \qquad (1)$$

where J_T is the instantaneous cost, e is the estimation error, $alpha_1$ is a scalar design parameter (see below) and y is the adaptive filter output.

Noting the expressions defining the transversal adaptive estimator's output and estimation error, at time index k, in terms of the reference input, x, and desired input, d :

$$y_k = W_k{}^T X_k$$
$$(X_k = [x_k,\ x_{k-1},\ ....]^T)$$

$$e_k = d_k - y_k$$

allows the gradient of the cost function to be written as :

$$d/dW\ (J_T) = alpha_1 y_k X_k - e_k X_k \qquad (2)$$

such that the update equation for the adaptive filter is given by:

$$W_{k+1} = W_k + \alpha[e_k X_k - alpha_1 y_k X_k] \qquad (3)$$

where α is the conventional scalar update speed parameter. Notice that when $alpha_1$ is zero, the update expression, 3, reduces to the familiar LMS algorithm.

The update algorithm above attempts to minimise the weighted sum of estimation error and estimation power. The value assigned to the parameter $alpha_1$ dictates the compromise between low error and low power; high $alpha_1$ (within the stability bounds defined below) forces the adaptive filter to a solution which favours low output power, at the expense of increased estimation error and vice versa. The update algorithm, 3, is not new and has been successfully applied to the adaptive control of acoustic systems (see, for example, [2]). The following analysis of the adaptive filter's behaviour, under the control of equation 3, is original.

It has been shown [3] that for certain classes of input signal and under certain sampling conditions, the behaviour of an adaptive estimator can be exactly described by an equivalent linear transfer function; a frequency domain ratio between estimation error and desired input.

CONSTRAINED OUTPUT POWER ADAPTIVE FILTERS

The equivalent transfer function analysis technique may be extended to the analysis of systems under the control of constrained output power update algorithms, such as equation 3 [4]. The details of the analysis are not included in this paper, which will focus upon the results.

The equivalent transfer function of an adaptive transversal estimator controlled by eqn. 3, under the following constraint upon the reference sequence:

$$X_n^T X_k = N r_{k-n} \qquad (4)$$

where N is the length of the adaptive filter's impulse response and r_{k-n} is the reference signal's autocorrelation, lag k-n, is given by (see [4]):

$$\frac{E(z)}{D(z)} = \frac{1 + \alpha\, alpha_1 NR(z)}{1 + \alpha(1 + alpha_1)NR(z)} \qquad (5)$$

where NR(z) is the Z transform of the reference autocovariance, minus the zero lag term (R(z) may be thought of as an estimate of the reference auto power spectrum).

Equation 5 reduces to the equivalent transfer function of the standard LMS controlled adaptive estimator when $alpha_1 = 0$. When $alpha_1$ is non-zero, equation 5 shows that the output variance constraint displaces both the pole and zero structure of the adaptive estimator. These changes with respect to the LMS control system influence both stability and steady state performance of the estimator, effects which are described, by example, below.

3. EXAMPLE - MINIMUM VARIANCE SINUSOIDAL ESTIMATION

As an example of the effects introduced by the output power constrained adaptive filter, 3, the minimum output variance synchronous estimation of a sinusoidal disturbance is considered. A more complicated example of the estimation of a generalised periodic sequence is presented in [4].

3.1 The Equivalent Transfer Function

If the reference sequence to a power constrained adaptive estimator is a cysoid of form :

$$x_k = A\cos(w_0 kT + Phase)$$

then, assuming that equation 4 is obeyed, the equivalent transfer function, 5, may be written as:

$$\frac{E(z)}{D(z)} = \frac{1 + \alpha\,\text{alpha}_1 \frac{LA^2}{2}\left[\frac{z\cos(w_0T) - 1}{z^2 - 2z\cos(w_0T) + 1}\right]}{1 + \alpha(1 + \text{alpha}_1)\frac{LA^2}{2}\left[\frac{z\cos(w_0T) - 1}{z^2 - 2z\cos(w_0T) + 1}\right]}$$

If the reference frequency is chosen as one quarter of the sample frequency, then the expression above simplifies considerably to:

$$\frac{E(z)}{D(z)} = \frac{z^2 + 1 - \alpha\,\text{alpha}_1 \frac{LA^2}{2}}{z^2 + 1 - \alpha(1 + \text{alpha}_1)\frac{LA^2}{2}} \qquad (6)$$

This example is used in the analyses of stability limits and steady state magnitude response, presented below.

3.2 STABILITY LIMITS

Given equation 6, it is possible to solve for those combinations of alpha and alpha_1 which place a system pole(pair) on the unit circle. This condition describes a stable bound on the values chosen for the update parameters.

The stable bounds for two values of filter length * reference power product, $LA^2/2$, are reported in Figure 1. Figure 1 shows that a compromise between alpha and alpha_1 must be accepted if the system is operated near the stable limit. This does not mean that a system which is adapting quickly, as a result of high alpha, cannot be subject to output power constraint; the power constraint is imposed by the <u>relative values</u> of alpha and alpha_1.

Note that, with suitable choice of alpha_1, the system can be stable outside the range of values of alpha shown in Figure 1, although interpretation of the system behaviour in these modes is difficult.

3.3 STEADY STATE RESPONSE

The steady state response of the constrained output power system is illustrated by evaluating the magnitude frequency response of equation 6 over the entire passband of the system.

Figure 2 (a-c) shows the effect of increasing the output power constraint, by increasing the value of alpha_1. In Fig. 2(a), the magnitude equivalent transfer function of the estimator with no output power constraint (i.e. $alpha_1=0$) is shown. The notch at the frequency of the reference sequence (the "cancelling notch" in noise control applications) is seen to reach down to a gain of 0. This is because the zeros of the unconstrained (LMS) estimator are exactly on the unit circle. As $alpha_1$ assumes non-zero values, the zeros of the equivalent transfer function are displaced from the unit circle. This is seen in Figs. 2(b,c), in which the cancelling notches do not reach 0, as a result of the increasingly stringent output power constraint.

Note that the "passband" of the magnitude transfer functions of the constrained power systems (Figs. 2(b,c)) is not strongly influenced by the power constraint. This is because the adaptive filter only has significant output at frequencies around that of the reference sequence cysoid (note that the adapting weights can perform a limited amount of "heterodyning"; the system is explicitly time-variant).

4. THE "LEAKY LMS" ALGORITHM

An alternative approach to obtaining an output power constrained adaptive estimator is to define a cost function from a sum of the estimation mean square error and the $(norm)^2$ of the adaptive filter's weights (shown below in "instantaneous" form):

$$J_T = e_k^2 + alpha_1 W_k^T W_k \qquad (7)$$

Although equation 7 is NOT formally a minimum output variance cost function, it practically behaves as one since, for many cases (given a fixed input sequence), filter output variance increases with the $(norm)^2$ of the weights and vice versa.

An adaptive filter built around the cost function, 7, has exactly the same update equation as the "leaky LMS" algorithm [1,4], and so provides an interesting new interpretation of the behaviour of the Leaky LMS algorithm:

$$W_{k+1} = W_k(1 - \alpha alpha_1) + \alpha e_k X_k \qquad (8)$$

An equivalent transfer function for the Leaky LMS algorithm can be derived [4]:

$$\frac{E(z)}{D(z)} = \frac{1}{1 + \alpha P(z)} \qquad (9)$$

where $P(z)$ is the Z transform of the reference autocovariance estimate, $X_k^T X_n$, minus the zero lag term, windowed in lag by the decaying exponential $(1 - \alpha\ alpha_1)^{n-1}$. This expression can, of course, be analysed for stability and steady state response using similar techniques to those employed in the discussion of the true minimum variance cost function above.

5. CONCLUDING REMARKS

The behaviour of a constrained power adaptive estimator has been shown to be amenable to exact analysis in certain deterministic situations. These signal environments, although limited, are of particular importance to the adaptive control of acoustic noise. The minimum output power concept also allows an interesting perspective upon the operation of the Leaky LMS algorithm.

6. REFERENCES

[1] B WIDROW and S D STEARNS, Adaptive Signal Processing, Prentice Hall, 1985.
[2] S J ELLIOTT, I M STOTHERS and P A NELSON, "A Multiple Error LMS Algorithm and its Application to the Active Control of Sound and Vibration", IEEE Trans. Acoust., Speech, Signal Processing, ASSP-35, p1423 (1987).
[3] P M CLARKSON and P R WHITE, "Simplified Analysis of the LMS Adaptive Filter Using Transfer Function Approximations", IEEE Trans. Acoust., Speech, Signal Processing, ASSP-35, p987 (1987)
[4] P DARLINGTON and G XU, "Equivalent Transfer Functions of Minimum Output Variance Mean Square Estimators", Submitted to: IEEE Trans. Acoust., Speech, Signal Processing, Dec. 1989

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Figure 1. Stable Bounds on Update Parameters for a Synchoronous Constrained Power Sinusoidal Estimator. (Reference Frequency = $f_s/4$)

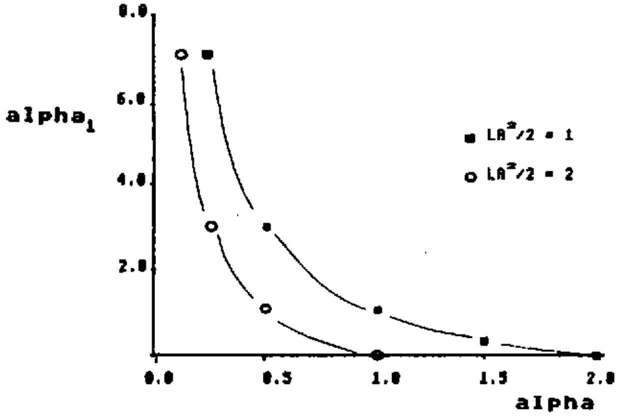


Figure 2 Magnitude Frequency Response of the Synchronous Constrained Power Sinusoidal Estimator. (Reference Frequency = $f_s/4$)

Figure 2(a)

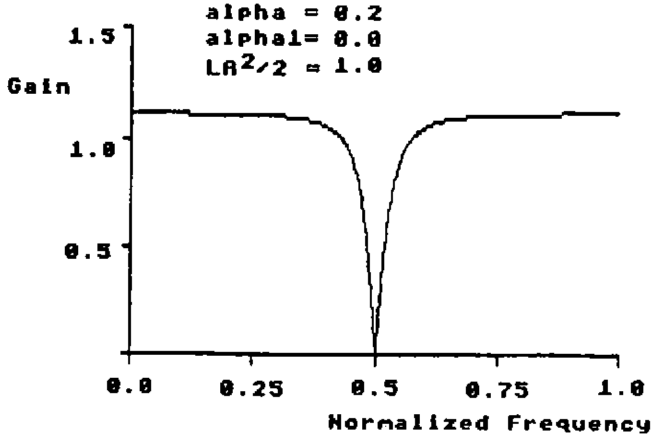


Figure 2(b)

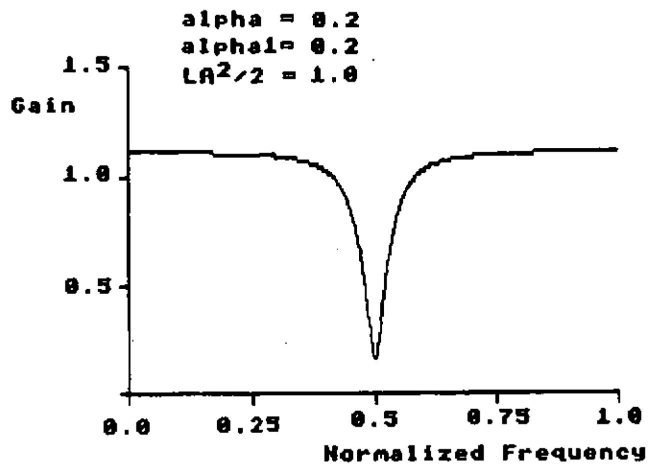
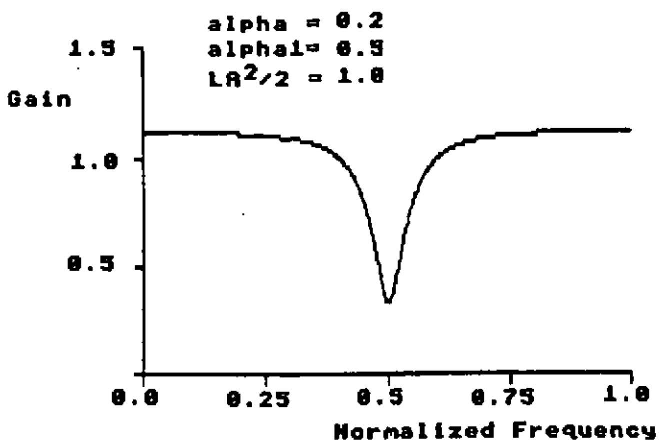


Figure 2(c)



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