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CAUSALITY, FILTERING AND PREDICTION IN ACTIVE NOISE CONTROL

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1. INTRODUCTION

The purpose of this paper is to introduce the concepts of filtering and prediction when considered in the context of active noise control. In order to illustrate the essence of these processes, active control mechanisms are presented for minimising the time-averaged squared pressure at a point on a spherical surface whose origin lies at the centre of a free field primary-secondary source pair. The secondary source is constrained to act causally with respect to the primary source.

At points on the surface which are closest to the secondary source, complete cancellation of the acoustic pressure is achievable since the secondary source needs only to act in response to the primary source and is therefore derived from it via a linear filter. At those remaining points closest to the primary source however, the optimal secondary source strength is derived from solving a predictor equation since now the secondary source must anticipate, and therefore predict the primary radiation at some time in the future from some current estimate of the primary source radiation statistics. The predictor equation arises quite naturally from any optimisation where causality is imposed and the source geometry requires that the secondary source needs to act in advance of the primary source [1]. The essence of the approach taken here has been presented by Nelson *et al* [2] in deriving the optimal causal relationship between a secondary and primary source when the total power output of the combination is minimised.

The predictor equation arising from this analysis is solved to obtain the optimal predictor where the primary radiation is obtained from filtered white noise and where the shaping filter is of second order. This analysis demonstrates that the degree of acoustic suppression attainable varies with the primary source autocorrelation function, the primary-secondary source separation and ultimately the point of suppression on the surface. The analysis further demonstrates that in the limiting case of white noise, the predictor equation assumes a trivial solution and is therefore redundant since white noise is totally uncorrelated with itself at any later time, and is therefore totally unpredictable.

2. THE GOVERNING EQUATION FOR THE CAUSALLY CONSTRAINED MINIMUM SQUARED PRESSURE

Consider a free-field primary-secondary source pair $q_p(t)$, $q_s(t)$ separated by a distance d, Fig. 1. Midway between the source pair is the centre of a sphere O of radius R on whose surface the squared pressure at any given point P is to be minimised.

The time-averaged squared pressure at P is given by

$$\langle p^2(R,\theta)\rangle = \lim_{T\to\infty} \frac{Z^2}{2T} \int_{-T}^{T} \left[\frac{q_p(t - r_p/c)}{r_p} + \frac{q_s(t - r_s/c)}{r_s}\right]^2 dt \qquad (1)$$

where Z is an impedance term $\rho_o c/4\pi$, $r_p^2 = R^2 + (d/2)^2 - Rd\cos\theta$, $r_s^2 = R^2 + (d/2)^2 + Rd\cos\theta$ and ρ_o and c are the density and sound speed of the medium. The integral is assumed to converge. If $q_s(t)$ is driven by $q_p(t)$ via some causal filter whose impulse response function is $h(\tau)$, then

$$q_s(t) = \int_0^\infty h(\tau)\, q_p(t - \tau) d\tau \qquad (2)$$

Substituting (2) into (1) and expanding yields

$$\langle p^2(R,\theta)\rangle = Z^2 \int_{-\infty}^{\infty} \left[\frac{q_p^2(t - r_p/c)}{r_p^2} + \frac{2q_p(t - r_p/c)}{r_p} \int_0^\infty h(\tau) \frac{q_p(t - r_s/c - \tau)}{r_s} d\tau\right.$$

$$\left. + \int_0^\infty \int_0^\infty h(\tau)h(\tau_1) \frac{q_p(t - r_s/c - \tau)q_p(t - r_s/c - \tau_1)}{r_s^2} d\tau d\tau_1\right] dt \qquad (3)$$

where the 'lim' formalism has been dropped.

Rearranging the order of operations allows the time-averaged squared pressure to be formulated in terms of the primary source autocorrelation function $R_{pp}(t_1, t_2)$, where

$$R_{pp}(t_1,t_2) = \overline{q_p(t_1)q_p(t_2)} = \int_{-\infty}^{\infty} q_p(t_1)q_p(t_1 + \tau)d\tau \qquad (4)$$

and $\tau = t_2 - t_1$.

If now the analysis is restricted to stationary random processes such that the source statistics are independent of any shift in the origin of time then

$$R_{pp}(t_1,t_2) = R_{pp}(t_2 - t_1)$$

and so $R_{pp}(\tau)$ represents the time-averaged or 'expected value' of the product $q_p(t)$ and itself $q_p(t + \tau)$ at a time τ later. Thus

$$\langle p^2(R,\theta)\rangle = Z^2\left[\frac{R_{pp}(0)}{r_p^2} + \frac{2}{r_p r_s}\int_0^\infty h(\tau)R_{pp}(\tau - (\frac{r_p - r_s}{c}))d\tau\right.$$

$$\left. + \frac{1}{r_s^2}\int_0^\infty\int_0^\infty h(\tau)h(\tau_1)R_{pp}(\tau - \tau_1)d\tau d\tau_1\right] \qquad (5)$$

The objective now is to minimise this quantity with respect to the optimal causal impulse response function $h_o(\tau)$.

Expanding $h(\tau)$ into the unknown optimal impulse response $h_o(\tau)$ plus some 'error' term $\epsilon h_e(\tau)$

$$h(\tau) = h_o(\tau) + \epsilon h_\epsilon(\tau). \qquad (6)$$

Any choice of the variational parameter ϵ effects an increase in $\langle p^2(R,\theta)\rangle$, and is therefore stationary about $\epsilon = 0$, i.e.,

$$(\frac{\partial\langle p^2(R,\theta)\rangle}{\partial\epsilon})_{\epsilon=0} = 0 \qquad (7)$$

Substituting (6) into (5), expanding in powers of ϵ and isolating the linear term according to (7) yields

$$\frac{2}{r_p r_s}\int_0^\infty h_\epsilon(\tau)R_{pp}(\tau - \frac{r_p - r_s}{c})d\tau$$

$$+ \frac{1}{r_s^2}\int_0^\infty\int_0^\infty (h_\epsilon(\tau)h_o(\tau_1) + h_\epsilon(\tau_1)h_o(\tau))R_{pp}(\tau - \tau_1)d\tau d\tau_1 = 0 \qquad (8)$$

Interchanging the dummy variables τ and τ_1 on the product $h_\epsilon(\tau_1)h_o(\tau)$ facilitates the factorization of the unknown error term $h_\epsilon(\tau)$ from (8) leaving the condition (9) on $h_o(\tau_1)$

$$\int_0^\infty h_o(\tau_1)R_{pp}(\tau - \tau_1)d\tau_1 = -(r_s/r_p)R_{pp}(\tau - (\frac{r_p - r_s}{c})) \text{ for } \tau > 0 \qquad (9)$$

since $h_\epsilon(\tau)$ is completely arbitrary and $\neq 0$.

This is a form of the well known Wiener-Hopf inhomogeneous integral equation for obtaining the least squares optimal impulse response function. Note that it is only the range of integration of τ_1 and the restriction on τ that ensures causality.

The interpretation of (9) is clear; given $q_p(t)$ we need to deduce $q_p(t - \eta)$ where $\eta = (r_p - r_s)/c$.

For $\eta \geq 0$, i.e., $r_p \geq r_s$, $h_o(\tau_1)$ is termed a filter since its role is to derive some current or past value from some present value of the input signal. For $\eta < 0$ or $r_s > r_p$ however, $h_o(\tau_1)$ must estimate the primary radiation at some time in the future from some current estimate of the source statistics and is therefore termed a predictor.

3. FILTERING AND PREDICTION

As the point of cancellation P moves over the surface of the sphere, both filtering and prediction regimes are encountered depending upon which of the source radiation travel times to the point P is the shorter. Each will now be considered in order to establish the degree of acoustic suppression that each affords.

3.1 The Filtering Domain $r_p \geq r_s$

For $r_p \geq r_s$, (9) has a simple solution. By inspection

$$h_o(\tau_1) = -\frac{r_s}{r_p}\,\delta\left(\tau_1 - \left(\frac{r_p - r_s}{c}\right)\right) \qquad (10)$$

from which we can deduce that

$$q_s(t) = -\frac{r_s}{r_p}\,q_p(t - r_p/c) \qquad (11)$$

Substituting (11) for $q_s(t)$ back into (1) demonstrates that the squared pressure at P may be driven to zero for all time independent of the nature of the primary source radiation

$$\langle p^2(\theta,R)\rangle = 0 \quad \text{for} \quad r_p \geq r_s \qquad (12)$$

This of course assumes that in practice an instantaneous measurement of the primary source strength is available and there are no further delays introduced by the detection and processing of this source strength in a practical system.

3.2 The Prediction Domain $r_s > r_p$

For $r_s > r_p$, the controlling secondary source must extract maximum information from the current primary radiation in an attempt to predict it at some time in the future. The predictability of the primary signal will clearly depend upon its autocorrelation function, a measure of how well the signal is linearly related to itself at some later time (or before).

The procedure for the calculation of the optimal predictor is given in standard texts (see [3]).

The transfer function of the optimal predictor $H_o(s)$ is calculated from

$$H_o(s) = H_1^{-1}(s)H_2(s) \tag{13}$$

where $H_1(s)$ is the transfer function of the shaping filter and $H_2(s)$ is the transfer function of the filter impulse response function shifted forwards in time by an amount η, i.e.,

$$H_o(s) = L\{h_o(t)\}$$

$$H_1(s) = L\{w(t)\}$$

$$H_2(s) = L\{w(t + \eta)\}$$

where L denotes the one-sided Laplace transform and $w(t)$ is the filter impulse response function

4. AN EXAMPLE OF AN OPTIMAL PREDICTOR

The procedure outlined above for obtaining the optimal predictor is now illustrated for primary source radiation obtained from white noise filtered through the filter most commonly occurring in sound and vibration phenomena, the second order resonator, i.e.,

$$H_1(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{14}$$

and

$$w(t) = \frac{\omega_n}{\sqrt{(1 - \zeta^2)}} e^{-\zeta\omega_n t} \sin \omega_n\sqrt{(1 - \zeta^2)}t \tag{15}$$

where ω_n and ζ are the filter's natural frequency and damping ratio respectively (see Figure 2). Constructing $w(t + \eta)$ and performing the inverse Laplace transform yields

$$H_2(s) = \frac{\omega_n}{\sqrt{(1 - \zeta^2)}} e^{-\zeta\omega_n\eta}$$

$$\times \left[\frac{\omega_n\sqrt{(1 - \zeta^2)}\cos \omega_n\sqrt{(1 - \zeta^2)}\eta + s \sin \omega_n\sqrt{(1 - \zeta^2)}\eta}{s^2 + 2\zeta\omega s + \omega^2}\right]$$

enabling derivation of the optimal predictor $H_o(s)$ which is thus given by

$$H_o(s) = H_2(s)H_1^{-1}(s)$$

$$= e^{-\zeta\omega_n\eta}\cos\omega_n\sqrt{(1-\zeta^2)}\eta + \frac{se^{-\zeta\omega_n\eta}\sin\omega_n\sqrt{(1-\zeta^2)}\eta}{\omega_n\sqrt{(1-\zeta^2)}} \qquad (16)$$

Thus the 'best' least squares prediction of $q_p(t)$ at a time η in the future $\dot{q}_p(t+\eta)$, is therefore

$$q_s(t) = \dot{q}_p(t+\eta)$$

$$= -\frac{r_p}{r_s}\left(e^{-\zeta\omega_n\eta}\cos\omega_n\sqrt{(1-\zeta^2)}\eta\, q_p(t) + \frac{e^{-\zeta\omega_n\eta}\sin\omega_n\sqrt{(1-\zeta^2)}\eta\,\dot{q}_p(t)}{\omega_n\sqrt{(1-\zeta^2)}}\right) \qquad (17)$$

Note that not only the current primary signal $q_p(t)$ but also its temporal derivative $\dot{q}_p(t)$ is employed in deriving the optimal prediction.

Substituting (17) for $q_s(t)$ in (1) and expanding gives the residual squared pressure at P after active control,

$$\langle p^2(\theta,R)\rangle = \frac{Z^2}{r_p^2}\Big[(1+\alpha^2)R_{pp}(0) - 2\alpha R_{pp}\left(\frac{r_s - r_p}{c}\right)$$

$$+ 2\alpha\beta R_{p\dot{p}}(0) + 2\beta R_{p\dot{p}}\left(\frac{r_s - r_p}{c}\right) + \beta^2 R_{\dot{p}\dot{p}}(0)\Big] \qquad \text{for } r_s > r_p \qquad (18)$$

where

$$\alpha = e^{-\zeta\omega_n\eta}\cos\omega_n\sqrt{(1-\zeta^2)}\eta, \qquad \beta = \frac{e^{-\zeta\omega_n\eta}\sin\omega_n\sqrt{(1-\zeta^2)}\eta}{\omega_n\sqrt{(1-\zeta^2)}}$$

and terms like $R_{p\dot{p}}\left(\frac{r_s - r_p}{c}\right)$ are formed from

$$R_{p\dot{p}}\left(\frac{r_s - r_p}{c}\right) = \langle q_p(t - r_p/c)\dot{q}_p(t - r_s/c)\rangle,$$

Note that in deriving (18) use has been made of the odd and anti-commutative properties of the cross correlation function, $R_{xy}(-\tau) = -R_{xy}(\tau)$ and $R_{yx}(\tau) = -R_{xy}(\tau)$.

Before proceeding to consider the problem for arbitrary η, consider the special cases of $\eta = 0$ and $\eta = -\infty$. For $\eta = 0$, $\alpha = 1$ and $\beta = 0$, therefore $\langle p^2(\pi/2,R)\rangle = 0$, corresponding to the case where the radiation travel times from both sources to the point P are identical. Thus providing the secondary source simultaneously mimics the primary source in antiphase then complete cancellation of the acoustic presure is possible.

For $\eta = -\infty$ however, $\alpha = \beta = 0$ so that $\langle P^2(\theta,\infty)\rangle = ((Z^2/r_p^2)R_{pp}(0))$, demonstrating the redundance of active control mechanisms when the secondary source is infinitely remote from the primary source.

Returning to the general solution (18), where the time averaged squared pressure at P is formulated in terms of $R_{pp}(\eta)$, $R_{p\dot{p}}(\eta)$ and $R_{\dot{p}\dot{p}}(\eta)$, then for a filter whose impulse response is $w(\tau)$ and whose input is white noise,

$$R_{pp}(\eta) = w(\tau) * w(\tau) \qquad (19)$$

where * denotes convolution.

For the second order filter (14) whose impulse response is

$$w(t) = \frac{\omega_n}{\sqrt{(1-\zeta^2)}} e^{-\zeta\omega_n t} \sin \omega_n\sqrt{(1-\zeta^2)}t \qquad (15)$$

then from (19)

$$R_{pp}(\eta) = \frac{\omega_n e^{-\zeta\omega_n\eta}}{8\zeta}\left[\cos \omega_n\sqrt{(1-\zeta^2)}\eta + \frac{\zeta}{\sqrt{(1-\zeta^2)}} \sin \omega_n\sqrt{(1-\zeta^2)}\eta\right] \qquad (20)$$

The remaining functions $R_{p\dot{p}}(\tau)$ and $R_{\dot{p}\dot{p}}(\tau)$ may be obtained from (20) via the relationships

$$R_{p\dot{p}}(\eta) = \frac{d}{d\eta} R_{pp}(\eta) \qquad (21)$$

and

$$R_{\dot{p}\dot{p}}(\eta) = -\frac{d^2}{d\eta^2} R_{pp}(\eta) \qquad (22)$$

Performing the differentiation

$$R_{p\dot{p}}(\eta) = -\frac{\omega_n e^{-\zeta\omega_n\eta}}{8\zeta}\left[\omega_n\sqrt{(1-\zeta^2)}\eta + \frac{\zeta^2\omega_n}{\sqrt{(1-\zeta^2)}}\right]\sin \omega_n\sqrt{(1-\zeta^2)}\eta \qquad (23)$$

and

$$R_{\dot{p}\dot{p}}(\eta) = \frac{\omega_n^2}{8\zeta} e^{-\zeta\omega_n\eta}\left[\cos \omega_n\sqrt{(1-\zeta^2)}\eta - \left(\frac{\zeta^3}{\sqrt{(1-\zeta^2)}} + \zeta\sqrt{(1-\zeta^2)}\right) \sin \omega_n\sqrt{(1-\zeta^2)}\eta\right] \quad * \text{ see below}$$

Therefore

$$R_{\dot{p}\dot{p}}(0) = \frac{\omega_n^2}{8\zeta} \qquad (24)$$

Note that the contribution to (18) from $2\alpha\beta R_{p\dot{p}}(0) = 0$, since $R_{p\dot{p}}(0) = 0$.

The expression (18) for the time-averaged pressure at P now reduces to

$$\langle p^2(\theta,R)\rangle = \frac{Z^2}{r_p^2}\left[(1+\alpha^2)\frac{\omega_n}{8\zeta} - 2\alpha R_{pp}\left(\frac{r_s - r_p}{c}\right) + 2\beta R_{p\dot{p}}\left(\frac{r_s - r_p}{c}\right) + \frac{\beta^2\omega_n^2}{8\zeta}\right] \qquad (25)$$

where $R_{pp}(\eta)$ and $R_{p\dot{p}}(\eta)$ are now completely defined.

5. DISCUSSION OF RESULTS

Expressions (12) and (25) for the time-averaged squared pressure at P are now evaluated as a fraction of the primary source pressure in the absence of the secondary source and as a function of the azimuthal angle θ around the sphere. Note that this is essentially a two dimensional problem since the difference in source travel times to the point P is independent of any polar angle Φ. Thus for a sphere of radius 200 m and a source separation of 100 m, the residual pressure at the point P after active control is shown in Fig. 3, where the filter centre frequency is 100 rad/s. Although these parameters are artificial in terms of attempting to model a realistic active noise control problem, they do serve to illustrate the essential features of the prediction process.

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*The authors have experienced some difficulty in evaluating $R_{\dot{p}\dot{p}}(0)$ using this method since $R_{\dot{p}\dot{p}}(\eta)$ becomes poorly conditioned for $\eta \to 0$. However, since the contribution from this term to the residual squared pressure is only second order in β, the errors arising from this simplification are believed to be negligible.

For $0 \leq \theta \leq \pi/2$ and $3\pi/2 \leq \theta \leq 2\pi$, P is closest to the secondary source and is therefore driven to zero for all time, (13). At those remaining angles, however, where P is closest to the primary source, the extent to which the acoustic pressure is suppressed is a function of the nature of the primary radiation which in turn is a function of the filter damping ratio ζ. For $\zeta = 1$, the filter is critically damped and therefore very nearly passes all frequency components such that the noise from the filter is very nearly 'white'. Such a signal is almost totally unpredictable as is manifest by the sudden rise in the residual squared pressure ratio to 1 at $\theta = \pi/2$. However, since the signal is only nearly white, the corners of the residual pressure profile are rounded. The absolute unpredictability of white noise may be demonstrated formally by returning to the Wiener equation (10) where for white noise $R_{pp}(\tau + ((r_s - r_p)/c))$ is a Dirac delta function centred on $\tau = -((r_s - r_p)/c)$ and is therefore 0 for $\tau > 0$. The optimal predictor $h_o(\tau)$ therefore assumes the unique, trivial solution $h_o(\tau) = 0$ from which $q_s(t) = 0$ indicating the redundance of active control methods for this type of signal.

As $\zeta \to 0$, the filter bandwidths becomes increasingly narrower from which the noise becomes increasingly deterministic. In the limit $\zeta = 0$, the filter bandwidth is sufficiently small so as to allow only a pure tone of the filter centre frequency to pass through, attenuating all other frequencies to insignificant levels. Since any purely sinusoidal signal is completely deterministic, the acoustic pressure may be driven to zero at all time.

Returning now to a more realistic free-space geometry where the source separation is 1 m and the radius of cancellation is 3 m. For this example, the primary source spectrum is altered by adjusting the filter centre frequency from 200 Hz to 3 kHz in 400 Hz increments, the filter damping ζ is set to 0.01, Fig. 4.

Again, complete pressure cancellation is achieved at points on the arc where $h_o(\tau)$ is a filter and less than perfect cancellation at those remaining points. Note that as the filter centre frequency increases, the residue pressure profile takes on more structure. This is because of increasingly larger variations of the correlation functions in (25) for the optimal prediction, compared with the wavelength from those frequencies contributing most significantly to the primary source spectrum, i.e., those nearest to the filter centre frequency, ω_n.

More importantly, however, is to note that even at $\theta = \pi$ where $h_o(\tau)$ must predict furthest in the future, on average, the squared pressure may still be driven to less than one fifth its original value.

6. CONCLUSIONS

The primary objective of this paper is to dispel the possible misconception that active control methods are redundant when the source geometry demands that an active secondary source neds to act in advance of a primary source whose sound field it is attempting to suppress. By example, this paper has shown that significant reductions in the time-averaged squared pressure may be achieved, even when the point of cancellation is appreciably closer to the primary source whose radiation is significantly 'random'

These results have particular significance to some practical active noise control implementations where real-time algorithms are employed which utilise the same least squares criterion for minimisation [4].

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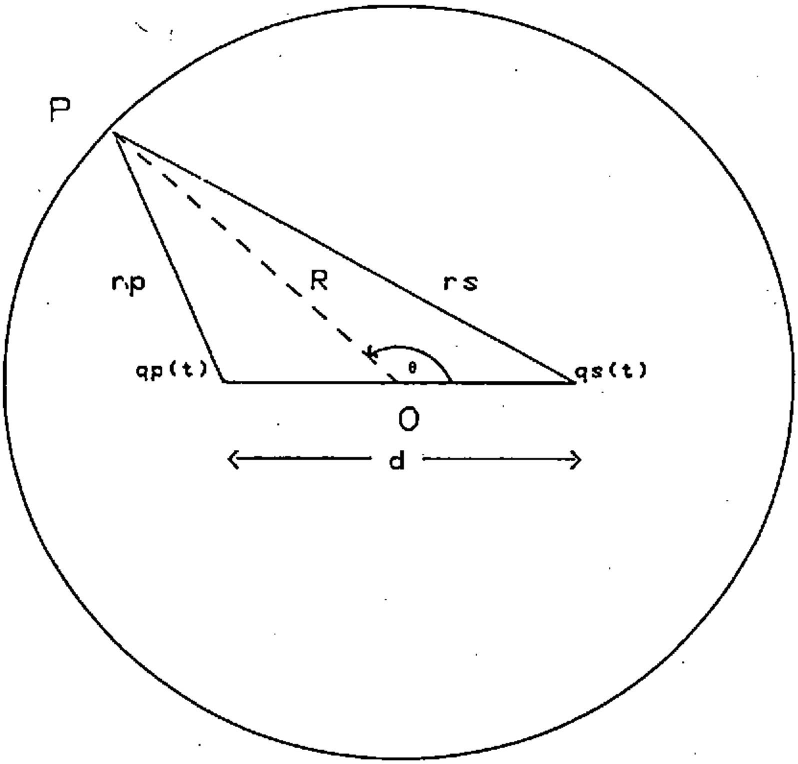


Fig 1

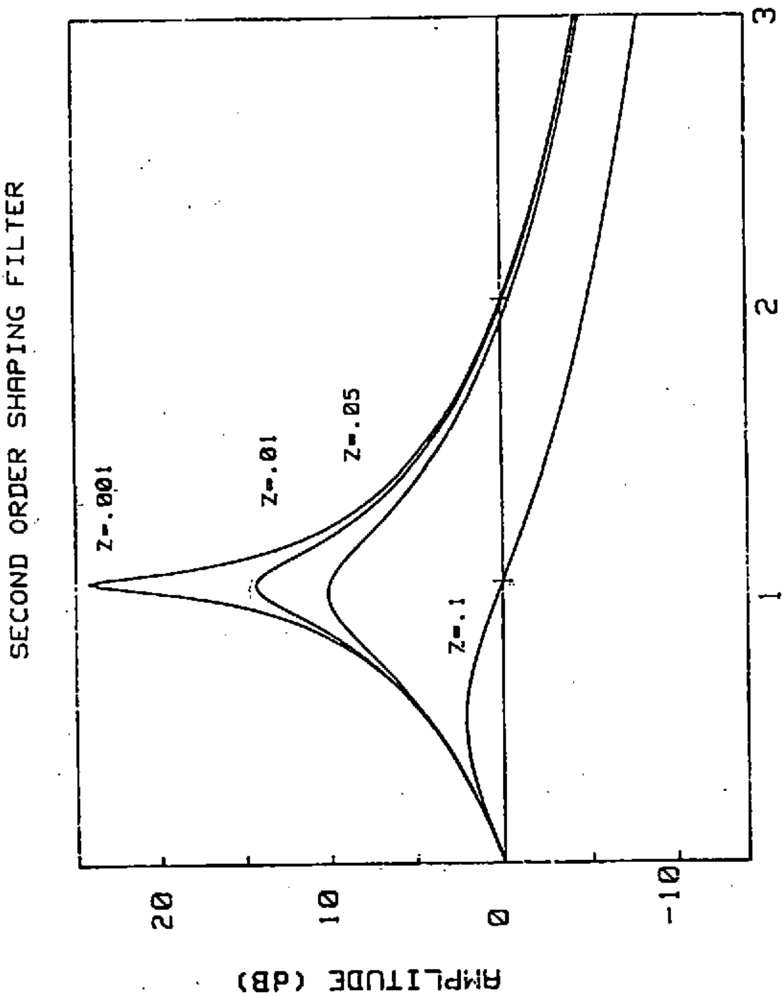
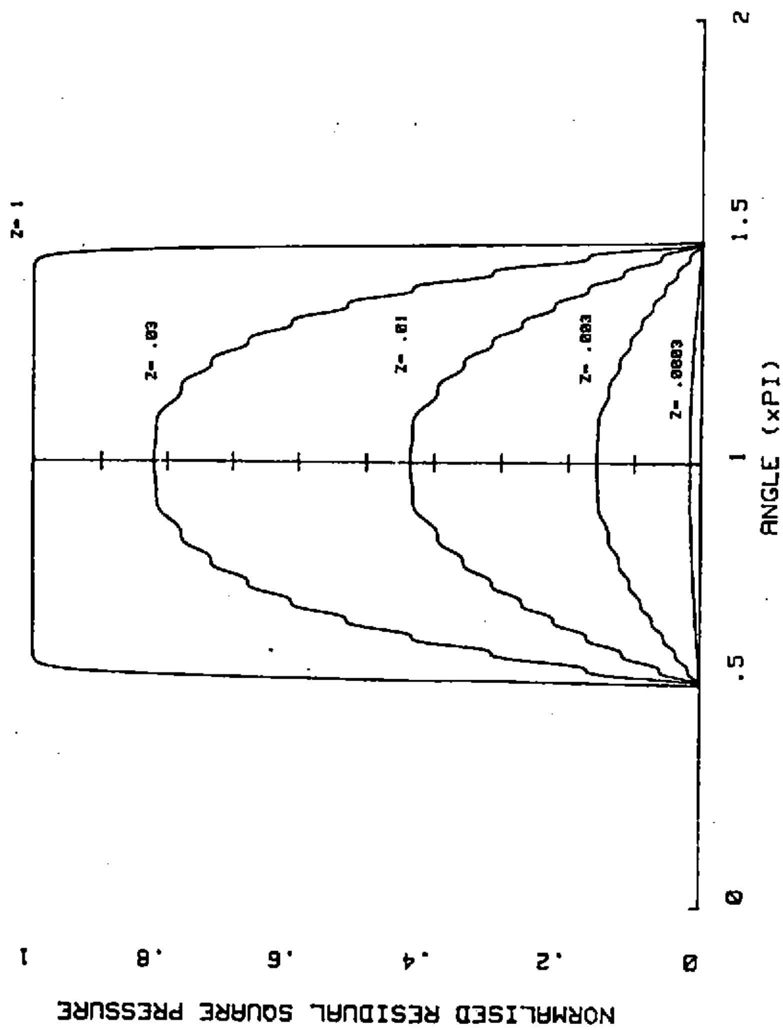

SECOND ORDER SHAPING FILTER
Z=.001
Z=.01
Z=.05
Z=.1
20
10
0
-10
AMPLITUDE (dB)
1
2
3
NORMALISED FREQUENCY

fig 2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Z= .03
Z= .01
Z= .003
Z= .0003
0
.5
1
1.5
2
ANGLE (xPI)
0
.2
.4
.6
.8
1
NORMALISED RESIDUAL SQUARE PRESSURE

Fig 3

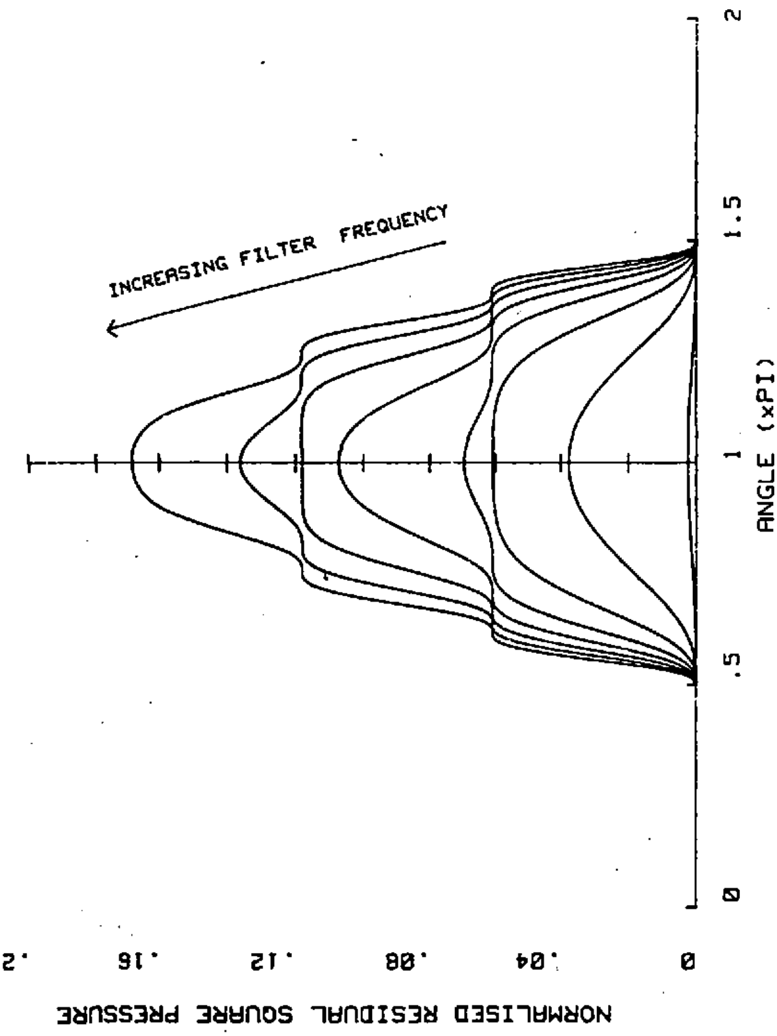
INCREASING FILTER FREQUENCY
ANGLE (xPI)
0
.5
1
1.5
2
NORMALISED RESIDUAL SQUARE PRESSURE
0
.04
.08
.12
.16
.2

Fig 4