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QUADRATIC OPTIMISATION PROBLEMS IN THE ACTIVE CONTROL OF FREE AND ENCLOSED SOUND FIELDS

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1. INTRODUCTION

It is the purpose of this paper to describe a theoretical framework which enables a clear quantification of the possible benefits that may be obtained from the control of sound fields by "secondary" sources. This framework is provided by the techniques of optimisation which form one of the foundations of modern control systems theory [1]. It is natural that these ideas should find their place in the study of the active control of sound, although this paper constitutes only the briefest introduction to their use. This type of analysis has been referred to only relatively recently by Roebuck [2] and Gaudefroy [3] and used by Piraux and Nayroles [4] in their treatment of the active control of three dimensional sound fields. The study of the active control of mechanical vibrations, however, is more amenable to the direct application of modern control systems theory, and extensive use has been made of these techniques in that area of research [5]. The objective of this work is to firstly illustrate some simple uses of optimisation theory, and secondly to demonstrate that some useful new results can be deduced, especially with respect to the active control of enclosed sound fields. It should be emphasised at the outset that the theory presented below assumes a perfect prior knowledge of the strength of the "primary" source distribution. Therefore the results presented enable quantification of the "best that can be done" by a given arrangement of secondary sources. These results are of course directly applicable to practical cases involving a periodic primary source where an explicit a-priori knowledge of the primary source strength is not required and use can be made of the waveform synthesis techniques developed by Chaplin [6].

2. THE MINIMUM POWER OUTPUT OF A PAIR OF MONOPOLE SOURCES

Consider a pair of simple sources, separated by a distance r, and each having a harmonic time dependence of the form $e^{j\omega t}$. Assume the "primary" source has a known fixed complex strength (volume velocity) q_p whilst the "secondary" source has a variable complex strength q_s. The total sound power output of the source combination can be written as

$$W = \tfrac{1}{2}\,\mathrm{Re}\{(p_{pp} + p_{ps})\, q_p^* + (p_{ss} + p_{sp})\, q_s^*\} \tag{1}$$

where p_{pp} and p_{ps} are the pressures produced at the primary source by the primary source and secondary source respectively. The pressures p_{ss} and p_{sp} are analagously defined. These pressures can in turn be related to the source strengths via the appropriate complex impedance. Thus $p_{pp} = Z_{pp}q_p$ and $p_{ps} = Z_{ps}q_s$ and so forth. Note that the principle of reciprocity (which applies under a wide range of conditions [7]) shows that $Z_{ps} = Z_{sp}$.

Equation (1) can be manipulated into the following form

$$W = q_s^* A q_s + q_s^* b + b^* q_s + c \tag{2}$$

where use of the principle of reciprocity enables the constants in this equation to be written as $A = \frac{1}{2}\,\mathrm{Re}\{Z_{ss}\}$, $b = \frac{1}{2}\,\mathrm{Re}\{Z_{ps}\}q_p$ and $c = \frac{1}{2}\,\mathrm{Re}\{Z_{pp}\}|q_p|^2$. This latter quantity is the power output W_p that would be produced by the primary source in the absence of the secondary source. The important feature of this equation is that it is a real quadratic function of the complex secondary source strength q_s. The quadratic nature of this equation can be further emphasised by expressing it in terms of the real and imaginary parts of the secondary source strength. Thus

$$W = Aq_{sR}^2 + 2b_R q_{sR} + Aq_{sI}^2 + 2b_I q_{sI} + c \tag{3}$$

where $q_{sR} = \mathrm{Re}\{q_s\}$, $q_{sI} = \mathrm{Im}\{q_s\}$ and $b_R = \mathrm{Re}\{b\}$, $b_I = \mathrm{Im}\{b\}$.
Figure 1 shows a plot of the function W against the variables q_{sR} and q_{sI}. The value of W is defined by a surface whose height above the (q_{sR}, q_{sI}) plane is determined for each particular combination of q_{sR} and q_{sI}. For a non-zero source separation distance 'r', W is a positive definite quadratic function of the variables q_{sR} and q_{sI} (i.e. W>O for all possible combinations of non-zero values of q_{sR} and q_{sI}). Furthermore, the function W will have a unique global minimum. That is there is only one particular combination of the variables q_{sR} and q_{sI} which defines the minimum value of W. This corresponds to the bottom of the "bowl" shaped surface illustrated in Figure 1. Thus there is

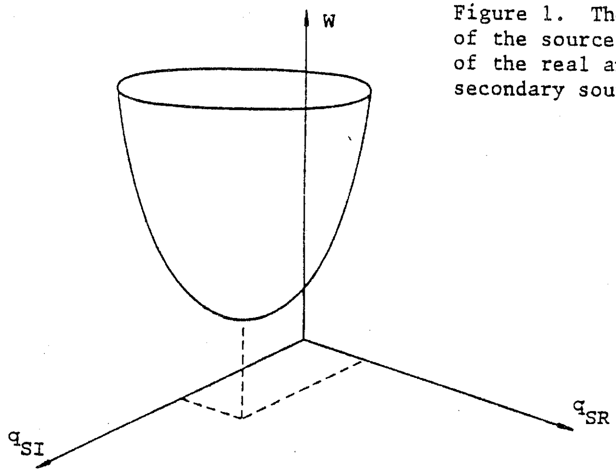

Figure 1. The total sound power output of the source combination as a function of the real and imaginery parts of the secondary source strength.

one particular complex value of the (modulus and phase) of the source strength q_s which minimises the total power output. This complex source strength is given when the derivative of W with respect to both q_{sR} and q_{sI} is equal to zero. This criterion is satisfied if

$$\frac{\partial W}{\partial q_{sR}} + j\,\frac{\partial W}{\partial q_{sI}} = 0 \tag{4}$$

Use of this result and equation (3) results directly in the solution for the optimal secondary source strength q_{so} which minimises the total power output of the source pair. This is given by

$$q_{so} = -A^{-1}b \tag{5}$$

and the corresponding minimum value of power radiated, W_o, is found by substitution of this result into equation (2), which yields

$$W_o = c - b^* A^{-1} b. \tag{6}$$

Note that equation (5) relates the optimal complex secondary source strength to the complex primary source strength. Now consider the case where the two monopoles are situated in a free field. Firstly observe that the complex impedance relating the strength of point simple source to the pressure at a distance r away can be written as

$$\frac{\omega^2\rho_o}{4\pi c_o}\left[\frac{\sin kr}{kr} + j\,\frac{\cos kr}{kr}\right]$$

where $k = \omega/c_o$ and ρ_o, c_o define the density and sound speed of the medium. Thus, since $(\sin kr/kr) \rightarrow 1$ as $r \rightarrow 0$, we have $\mathrm{Re}\{Z_{ss}\} = (\omega^2\rho_o/4\pi c_o)$ and $\mathrm{Re}\{Z_{ps}\} = (\omega^2\rho_o/4\pi c_o)\,(\sin kr/kr)$ which enables equation (5) to be written as

$$q_{so} = -\left[\frac{\sin kr}{kr}\right] q_p \tag{7}$$

Since $(\sin kr/kr)$ is real then the optimal secondary source strength must be π out of phase (or in phase) with the primary source strength in order to minimise the total power output (a result also deduced by Ffowcs Williams [8]). Note, however, that the magnitude of the optimal secondary source strength is dependent on the source separation distance relative to the wavelength at the frequency of interest. It is also useful to compare the minimum value of power radiated W_o with the power W_p radiated by the primary source alone

Division of equation (6) by W_p results in

$$W_o/W_p = 1 - (\sin kr/kr)^2 \tag{8}$$

QUADRATIC OPTIMISATION PROBLEMS IN THE ACTIVE CONTROL OF FREE AND ENCLOSED SOUND FIELDS

The corresponding fractional reduction in sound power output is plotted in Figure 2. This clearly demonstrates that no appreciable reduction in source power output can be achieved if the secondary source is separated from the primary source by a distance which exceeds half a wavelength at the frequency of interest.

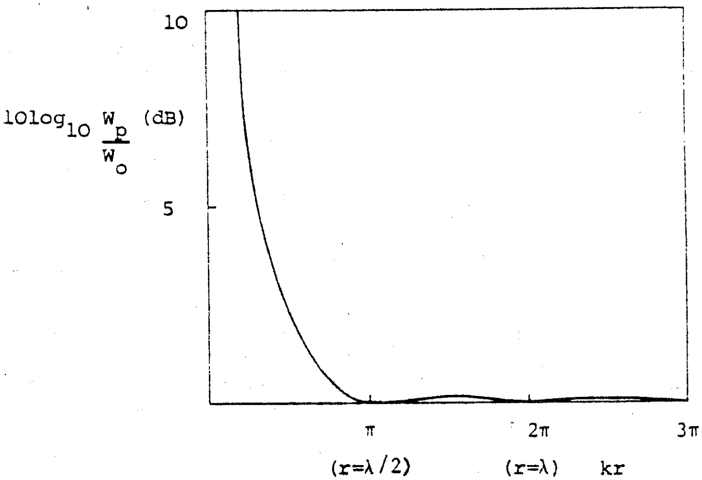

Figure 2. The maximum reduction in total sound power radiated as a function of normalised primary/secondary source separation distance.

3. THE MINIMUM POWER OUTPUT OF A NUMBER OF MONOPOLE SOURCES

It will now be demonstrated that the above analysis can be naturally extended to deal with an arbitrary number of both "primary" and "secondary" simple sources. The total power output of these sources can be written as

$$W = \tfrac{1}{2}\,\mathrm{Re}\{(\underline{p}_{pp} + \underline{p}_{ps})^T \underline{q}_p^* + (\underline{p}_{ss} + \underline{p}_{sp})^T \underline{q}_s^*\} \qquad (9)$$

where $\underline{q}_p$ is the vector of complex primary source strengths and $\underline{q}_s$ is the vector of complex secondary source strengths. The vectors $\underline{P}_{pp}$ and $\underline{P}_{ps}$ define the pressures produced at the primary sources by the primary sources and secondary sources respectively and similarly for $\underline{P}_{ss}$ and $\underline{P}_{sp}$. Note that the symbol T denotes the transpose of a vector. The vectors of complex pressures can be related to the complex source strength vectors via complex impedance matrices. Thus $\underline{P}_{pp} = \underline{Z}_{pp}\underline{q}_p$ and $\underline{P}_{ps} = \underline{Z}_{ps}\underline{q}_s$ and so forth. The principle of reciprocity can again be used and in this instance it follows that the matrices $\underline{Z}_{pp}$ and $\underline{Z}_{ss}$ are square symmetric ($\underline{Z}_{pp}^T = \underline{Z}_{pp}$, $\underline{Z}_{ss}^T = \underline{Z}_{ss}$) and also that $\underline{Z}_{ps} = \underline{Z}_{sp}^T$. Equation (9) can be manipulated into the form

$$W = \underline{q}_s^H \underline{A} \, \underline{q}_s + \underline{q}_s^H \underline{b} + \underline{b}^H \underline{q}_s + c \qquad (10)$$

Where the relationships deduced from the principle of reciprocity can be used to show that the matrix $\underline{A}$ is given by $\frac{1}{2}\mathrm{Re}\{\underline{Z}_{ss}\}$, the vector $\underline{b}$ is given by $\frac{1}{2}\mathrm{Re}\{\underline{Z}_{sp}\}\underline{q}_p$ and the scalar constant c is $\underline{q}_p^H (\frac{1}{2}\mathrm{Re}\{\underline{Z}_{pp}\})\underline{q}_p$.
Note that the symbol H denotes the complex conjugate of a vector transpose (i.e. $\underline{q}_s^H = (\underline{q}_s^*)^T$). The scalar constant c in this case is equal to W_p, the total net power radiated by the collection of primary sources in the absence of any secondary sources. The analogy between equation (10) and equation (2) is clear, the latter of course being a special case of the former. Furthermore, the properties of equation (10) are also best described in terms of those of equation (2). Firstly, equation (10) shows that the total power is again a quadratic function of the strengths of the secondary sources. (This can be demonstrated by expansion of equation (10) in terms of the real and imaginery parts of the components of the secondary source strength vector, the case of two secondary sources being the simplest example). The function W can thus again be thought of as describing a bowl shaped surface similar to that illustrated in Figure 1, but in this case it is a hypersurface whose "height" is defined as a function of the number of variables equal to twice the number of secondary sources. Such a surface defies illustration except in the two variable case shown in Figure 1. Secondly, however, and most importantly, this surface again has a unique global minimum corresponding to the one particular combination of secondary source strengths which defines the bottom of the bowl. This minimum value is defined when the derivative of W with respect to the real and imaginary parts of all the components of the secondary source strength vector is set equal to zero. This condition can be written as

$$\frac{\partial W}{q_{sR}} + j \frac{\partial W}{\partial q_{sI}} = 0 \qquad (11)$$

Differentiation of equation (10) then yields the solution for the optimal vector of secondary source strengths which minimises the total power radiated. This is given by

$$\underline{q}_{so} = -\underline{A}^{-1}\underline{b} \qquad (12)$$

Substitution of this result into equation (10) yields the corresponding minimum value of W which is given by

$$W_o = c - \underline{b}^H \underline{A}^{-1} \underline{b} \qquad (13)$$

The analogy of these results with those for the simple scalar case (equations (5) and (6)) is clear. It is also interesting to compute the maximum reduction in power output that can be achieved given a particular arrangement of primary and secondary sources. Consider the case of a single primary source and a number of secondary sources. The optimal sector of secondary source strengths given by equation (12) in this case is

$$\underline{q}_{so} = - \left[\mathrm{Re}\{\underline{Z}_{ss}\}\right]^{-1}\mathrm{Re}\{\underline{Z}_{sp}\}q_p \qquad (14)$$

where q_p is the complex strength of the single primary source. This shows that for minimum total power output, the secondary source strengths must all be π out of phase (or in phase) with the primary source strength. The corresponding ratio of the minimum power output W_o to the power output W_p of the primary source alone follows from equation (13) (noting that $\underline{Z}_{pp}$ reduces to a scalar);

$$\frac{W_o}{W_p} = 1 - \frac{\left[\mathrm{Re}\{\underline{Z}_{sp}\}\right]^T \left[\mathrm{Re}\{\underline{Z}_{ss}\}\right]^{-1} \left[\mathrm{Re}\{\underline{Z}_{sp}\}\right]}{\mathrm{Re}\{Z_{pp}\}} \qquad (15)$$

for the case of the primary and secondary sources in a free field this evidently depends on only the relative locations of the sources and has been evaluated for a number of secondary sources placed in an icosahedral array surrounding a central primary source (Figure 3). Figure 4 shows the result

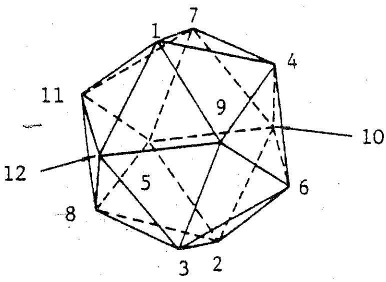

Figure 3. The positions of the point secondary sources placed in an icosahedral array surrounding the single point primary source at the centre of the icosahedron.

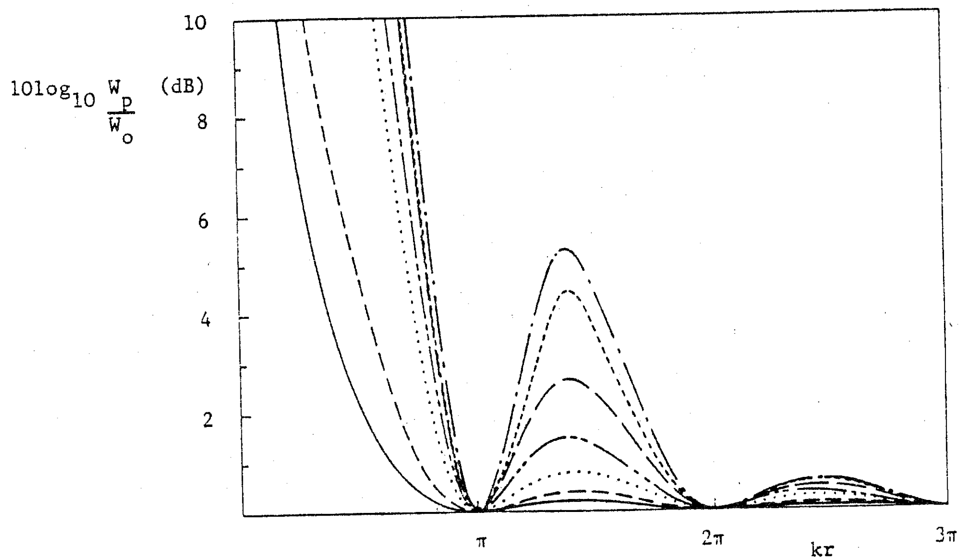

Figure 4. Maximum reductions in total sound power radiated for the source arrangement shown in Figure 3. ——1 source, — — —2 sources, ... 4 sources, —— — — —— 6 sources, ———— ——8 sources, ----- 10 sources, —— — —— 12 sources.

of this calculation as a function of the distance of the secondary sources from the primary source and the number of secondary sources placed in the positions shown. (Note that several of the sources shown in Figure 3 were moved slightly from the positions shown to avoid conditioning problems in the matrix inversion). The conclusion drawn from Figure 3 is that substantial reductions in sound power output are again only produced when all the secondary sources are placed within half a wavelength of the primary source at the frequency of interest. There are however, considerable gains to be had by increasing the number of secondary sources placed within this distance from the primary source.

4. THE MINIMUM ACOUSTIC POTENTIAL ENERGY IN AN ENCLOSURE

It will now be demonstrated that the acoustic potential energy in an enclosed sound field can be expressed as a positive definite quadratic function of the complex strengths of a number of secondary sources introduced into the enclosure. The acoustic potential energy is given by

$$E_p = \frac{1}{4\rho_o c_o^2} \int_V |p|^2 \, dV \qquad (16)$$

and is thus the relevant quantity to minimise if global reductions are sought in the magnitude of the pressure fluctuations in the sound field. The complex pressure p can be expressed in terms of a truncated series of normal modes having a vector of complex amplitudes $\underline{a}$ and a vector of real normalised characteristic functions ψ such that $p = \psi^T \underline{a}$. It follows from the orthogonality property of the normal modes that equation (16) reduces to

$$E_p = \frac{V}{4\rho_o c_o^2} \underline{a}^H \underline{a} \tag{17}$$

Now note that each of the complex mode amplitudes can be considered to consist of a contribution from some unspecified "primary" source distribution (due to wall vibration or volume sources for example) plus the contribution from a series of point secondary sources deliberately introduced to control the sound field. Thus

$$\underline{a} = \underline{a}_p + \underline{B}\underline{q}_s \tag{18}$$

where $\underline{a}_p$ is the vector of complex mode amplitudes produced by the primary source distribution only and the matrix $\underline{B}$ quantifies the extent to which each mode is excited by each secondary source. Combination of equations (17) and (18) can be written in the now familiar quadratic form

$$E_p = \underline{q}_s^H \underline{A}\, \underline{q}_s + \underline{q}_s^H \underline{b} + \underline{b}^H \underline{q}_s + c \tag{19}$$

where in this case the matrix $\underline{A} = \left(V/4\rho_o c_o^2\right)\underline{B}^H\underline{B}$, the vector $\underline{b} = \left(4\rho_o c_o^2\right)\underline{B}^H\underline{a}_p$ and the scalar constant $c = \left(V/4\rho_o c_o^2\right) \underline{a}_p^H \underline{a}_p$ which is E_{pp}, the acoustic potential energy in the enclosure due to the primary source alone. The utility of this expression again stems from the fact that this function has a unique global minimum, and that for a given primary source distribution and a given number and location of secondary sources, there is a unique combination of secondary source strengths which minimises this function. It therefore provides an ideal framework for more detailed consideration of what reductions are <u>possible in principle</u> to achieve by way of the active control of the sound levels in an enclosure. This analysis has been used [9] to deduce the reductions that are possible for the particular case of a single point primary source surrounded by a number of point secondary sources in an enclosure, when the wavelength at the frequency of interest is much shorter than the dimensions of the enclosure. That is, at the frequency of interest, there is a high density of acoustic modes contributing to the sound field. It is shown [9] in this case that the ratio of the minimum value of acoustic potential energy, E_{po}, to that produced by the primary source alone (E_{pp}), is predicted exactly by equation (15) above. That is

$$\frac{E_{po}}{E_{pp}} = \frac{W_o}{W_p} \tag{20}$$

and the reductions in the acoustic potential energy in the enclosure are exactly equal to the possible reductions in the net source power output in the free field case. This is not surprising in view of the inherent relationship between the source power output and the energy of the enclosed sound field in the high frequency limit. Equivalent conclusions thus also result regarding the effectiveness of the relative locations of primary and secondary sources. This analysis remains to be applied to the case of the more complicated "primary" source distributions associated with problems of practical interest. However, some other interesting conclusions can be drawn from consideration of enclosed sound fields excited at frequencies where there is a low density of acoustic modes.

5. CONCLUSIONS

A theory has been presented which uses the techniques of optimisation for the analysis of various problems associated with the active control of sound. The theory assumes a perfect prior knowledge of the strength of the primary source distribution and thus enables an unequivocal deduction of the "best possible reduction" that can be achieved by active techniques. The application of the theory to the case of a single point primary source has shown that substantial reductions in the net acoustic power radiated can only be achieved provided secondary sources are placed within a distance of half a wavelength at the frequency of interest. Identical conclusions result regarding the reductions in the acoustic potential energy that can be achieved in the case of a point primary source exciting an enclosed sound field having a high density of normal modes.

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