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LINEAR LEAST SQUARES ESTIMATION PROBLEMS IN THE ACTIVE CONTROL OF STATIONARY RANDOM SOUND FIELDS

P.A. Nelson, J.K. Hammond and S.J. Elliott

Institute of Sound and Vibration Research, University of Southampton, Southampton, England.

1. INTRODUCTION

A useful means of analysing problems in the active control of sound and vibration is provided by quadratic optimisation theory. This enables the unambiguous quantification of the degree to which various cost functions (such as acoustic squared pressure, power or energy) can be reduced by the action of secondary sources. This approach uses an analysis in the frequency domain which has proved extremely useful in the treatment of problems such as the active control of enclosed sound fields and the active suppression and absorption of acoustic radiation. A summary of this approach and its application to several examples is presented in reference [1]. The major, well-recognised, drawback to this method of analysis is that there is no constraint on the causality of the solutions produced. Thus, in many instances in order to achieve optimal results, secondary sources must produce outputs in anticipation of primary source outputs. This approach cannot therefore be used with any generality for treating practical problems involving the control of random sound fields.

The work presented here aims to complement this frequency domain theory with a formulation in the time domain which does allow the constraint of causality to be imposed upon the action of the secondary sources. The formulation used does not, however, enable results to be deduced as easily as those deduced by working in the frequency domain. The approach adopted is based on the classical techniques of linear estimation theory and results in the derivation of a Wiener-Hopf integral equation which must be satisfied by the impulse response of the optimal causal controller. The derivation of this equation is presented below for the particular problem posed by the feed-forward control of random sound fields. Some comments are firstly made regarding the general problem formulation and then a simple example is presented for the use of the technique when the integral equation can be solved analytically. The example demonstrates the importance of the statistical predictability of the sound field being dealt with and its influence on the maximum achievable performance limits of active control systems dealing with random sounds.

2. FORMULATION OF THE GENERAL OPTIMAL FILTERING PROBLEM

Figure 1 is a block diagram representation of the general feed-forward active control problem. We assume that there are P primary sources, whose sound fields are detected by K detection sensors. These signals constitute the elements of the vector $\underline{x}(t)$ which pass through a matrix $\underline{G}$ of filters whose output vector $\underline{y}(t)$ is fed to M secondary sources. The possibility of these output signals being detected by the detection sensors is represented by the matrix $\underline{F}$ of feedback filters. The secondary sources produce signals $-\hat{\underline{d}}(t)$ at L error sensors. The primary sources also produce signals $\underline{d}(t)$ at the L error sensors. The interference of the primary and secondary source fields produces a vector $\underline{e}(t)$ of error signals. We can allow for the contamination of the signals $\underline{x}(t)$ and $\underline{d}(t)$ by the inclusion of noise signals $\underline{n}_1(t)$ and $\underline{n}_2(t)$. The problem is to determine the matrix $\underline{G}$ which minimises a weighted sum of squared error signals. This problem has been dealt with by using an analysis in the frequency domain [2] but this of course has no constraint on the causality of the elements of the matrix $\underline{G}$. Here we attempt to determine the elements of this matrix when they are constrained to be causal filters and we seek to minimise the expected value of the weighted sum of squared error time histories.

Firstly consider the problem of the feedback represented by the matrix $\underline{F}$. The approach we shall take is to formulate the problem in terms of the matrix $\underline{H}$ of filters which is formed from the combination of $\underline{G}$ and $\underline{F}$. We seek to determine the elements of $\underline{H}$ which minimise the required cost function such that the problem reduces to that illustrated in Figure 2. Having determined the optimal causal value of $\underline{H}$, given by $\underline{H}_0$, we can then proceed to determine the optimal value of $\underline{G}$ using manipulations in the frequency domain. This is illustrated in Figure 3 which shows the relationship between the optimal $\underline{G}_0$ and the filters $\underline{H}_0$ and $\underline{F}$. If $\underline{H}_0$ is causal, and since in any physical problem $\underline{F}$ must be causal, then the filter $\underline{G}_0$ must be causal. This does not, however, guarantee the stability of $\underline{G}_0$ and in any practical circumstances this will have to be ensured by the elimination of the destabilising influence of $\underline{F}$ (by, for example, the use of directional detection sensors and directional secondary sources).

We can now show that the problem of determining $\underline{H}_0$ can be cast in the form of a classical estimation problem. This is most easily accomplished by using the property of linear systems that the output produced by two systems in series is the same for a given input even if the order of the operation of the systems is reversed. Figure 4 summarises the approach that we take here. Thus the signal produced at the ℓ'th error sensor is written in terms of the outputs of the elements of the matrix $\underline{H}$. The justification for this reversal of the transfer functions in the block diagram of Figure 2 is given by working in the frequency domain. Thus we express the signals $\underline{y}$ and $\hat{\underline{d}}$ as

$$\underline{y} = \underline{H}\,\underline{x}, \qquad \hat{\underline{d}} = \underline{C}\,\underline{y} \tag{1}$$

where these can be written in terms of the columns $\underline{h}_K$ of $\underline{H}$ and the rows $\underline{c}_\ell^T$ of the matrix $\underline{C}$, such that

$$\underline{y} = [\underline{h}_1\ \underline{h}_2\ \ldots\ \underline{h}_K]\underline{x}, \qquad \hat{\underline{d}} = \begin{bmatrix}\underline{c}_1^T\\ \underline{c}_2^T\\ \vdots\\ \underline{c}_L^T\end{bmatrix}\underline{y} \tag{2}$$

Now considering the signal at the ℓ'th error sensor shows that this can be written as

$$\hat{d}_\ell = \underline{c}_\ell^T\underline{y} = \underline{c}_\ell^T(\underline{h}_1x_1 + \underline{h}_2x_2\ \ldots\ \underline{h}_Kx_K) \tag{3}$$

and since this is a scalar, we can transpose each of the contributions in equation (3) to give

$$\hat{d}_\ell = \underline{h}_1{}^T\underline{c}_\ell x_1 + \underline{h}_2{}^T\underline{c}_\ell x_2 \; \ldots \; \underline{h}_K{}^T\underline{c}_\ell x_K \qquad (4)$$

We can thus express the signal $\hat{\underline{d}}_\ell$ in terms of the vector $\underline{h}$ comprising the columns of $\underline{H}$ and the vector $\underline{r}_\ell$, where we thus define

$$\hat{d}_\ell = \underline{h}^T\underline{r}_\ell, \qquad \underline{h} = \begin{bmatrix}\underline{h}_1\\ \underline{h}_2\\ \vdots\\ \underline{h}_K\end{bmatrix}, \qquad \underline{r}_\ell = \begin{bmatrix}\underline{c}_\ell x_1\\ \underline{c}_\ell x_2\\ \vdots\\ \underline{c}_\ell x_K\end{bmatrix} \qquad (5)$$

This then demonstrates that the expression for the signal $\hat{\underline{d}}_\ell(t)$ can be written in terms of a convolution of the vector of signals $\underline{r}_\ell(t)$ with the vector of causally constrained impulse responses $\underline{h}(t)$ such that

$$\hat{d}_\ell(t) = \int_0^\infty \underline{h}^T(\tau_1)\underline{r}_\ell(t - \tau_1)d\tau_1 \qquad (6)$$

where $\underline{h}(t) = 0$ for $t < 0$. The principle of transfer function reversal described here was used in reference [3]. In that case the signals $r_{\ell mk}$ were referred to as "filtered reference signals" and we shall adopt the same nomenclature here.

3. THE DERIVATION OF A WIENER-HOPF INTEGRAL EQUATION

We shall now demonstrate that formulating the problem in this way enables the derivation of an integral equation which must be satisfied by the optimal value of the vector of impulse responses $\underline{h}(t)$ which minimises are chosen cost function. The cost function can be expressed as

$$J = E[\underline{e}^T(t)\underline{\underline{W}}\underline{e}(t)] = \sum_{\ell=1}^{L} w_\ell(d_\ell(t) - \hat{d}_\ell(t))^2 \qquad (7)$$

where $\underline{W}$ is a diagonal matrix of weighting factors whose elements are w_ℓ. Although it is clear that in the case of sound field control we can interpret the error signals $\underline{e}(t)$ as, for example, the pressures produced at a number of microphones, it can also be shown that problems of, for example, minimising power and energy can also be formulated in this way. In order to deduce the optimal value of $\underline{h}(t)$ which minimises this function, we follow the standard technique presented for the scalar case in reference [4]. A full description of the derivation is given in the Appendix. One proceeds by first substituting equation (6) into equation (7) to produce an expression for J in terms of the impulse response vector $\underline{h}(\tau_1)$. One then assumes that $\underline{h}(\tau_1)$ can be expressed as the sum of an optimal value $\underline{h}_o(\tau_1)$ plus another arbitrary vector of impulse responses $\underline{h}_\epsilon(\tau_1)$ such that

$$\underline{h}(\tau_1) = \underline{h}_o(\tau_1) + \epsilon\underline{h}_\epsilon(\tau_1) \qquad (8)$$

where ϵ is an arbitrary real parameter. The problem then reduces to demonstrating that J is increased above its minimum value J_o for any choice of the parameter ϵ and the impulse response vector $\underline{h}_\epsilon(\tau_1)$. As shown in the Appendix, a necessary and sufficient condition for this to be true is given by

$$\underline{R}_{dr}(\tau_1) - \int_0^\infty \underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_o(\tau_2)d\tau_2 = 0 \qquad \tau_1 > 0 \qquad (9)$$

where the vector of cross correlations $\underline{R}_{dr}(\tau_1)$ and the matrix of auto-correlations $\underline{R}_{rr}(\tau_1 - \tau_2)$ are given by

$$\underline{R}_{dr}(\tau_1) = \sum_{\ell=1}^{L} w_\ell E[d_\ell(t)\underline{r}_\ell(t - \tau_1)] \qquad (10)$$

$$\underline{R}_{rr}(\tau_1 - \tau_2) = \sum_{\ell=1}^{L} w_\ell E[\underline{r}_\ell(t - \tau_1)\underline{r}_\ell^T(t - \tau_2)] \qquad (11)$$

Equation (9) is the principal result of this paper. Implicit in the derivation is that the random signals considered are stationary and that the impulse response $\underline{h}(\tau_1)$ is causal, i.e., $\underline{h}(\tau_1) = 0$ for $\tau_1 > 0$. It is also shown in the Appendix that choosing the optimal value of the impulse response vector which satisfies equation (9) results in the minimum value of the cost function given by

$$J_o = R_{dd}(0) - \int_0^\infty \underline{h}_o^T(\tau_1)\underline{R}_{rd}(\tau_1)d\tau_1 \qquad (12)$$

where $R_{dd}(0) = \sum_{\ell=1}^{L} w_\ell E[d_\ell^2(t)]$. Note that equation (12) can also be written in the form

$$J_o = R_{dd}(0) - \sum_{\ell=1}^{L} w_\ell E[d_\ell(t)\hat{d}_{\ell o}(t)]. \qquad (13)$$

Determination of the optimal vector of impulse response functions thus reduces to the solution of the matrix Wiener-Hopf integral equation given by equation (9). As we shall show below in the single channel case, analytical techniques can be used to solve this equation in some cases although in the multi-channel case numerical methods will generally be necessary for the solution to the equation. Note that one such numerical method is provided by Elliott's stochastic gradient algorithm presented in reference [3]. The extension of this algorithm to the case where multiple primary sources are present follows easily from the above formulation.

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4. A SINGLE CHANNEL EXAMPLE; THE MINIMISATION OF MEAN SQUARED ACOUSTIC PRESSURE

As an illustration of the solution to the Wiener-Hopf equation, we consider the problem depicted in Figure 5. This is the very simple active control problem of minimising the squared pressure at a radial distance r_p from a primary monopole source of strength $q_p(t)$ using a secondary source of strength $q_s(t)$ where r_s represents the radial distance from the secondary source to the observation point. The equivalent block diagram of the problem is shown in Figure 5(b) and in Figure 5(c) the problem is depicted with the transfer functions H and C reversed, thus enabling the definition of the filtered reference signal r(t). It follows automatically that the Wiener-Hopf equation that must be solved is given by

$$\left(\frac{r_s}{r_p}\right)R_{pp}\left(\tau_1 - \frac{r_p - r_s}{c_o}\right) + \int_0^\infty R_{pp}(\tau_1 - \tau_2)h_o(\tau_2)d\tau_2 = 0 \qquad \tau_1 > 0 \quad (14)$$

where R_{pp} is the autocorrelation function of the primary source strength time derivative.

The solution to equation (14) is crucially dependent upon the value of the time difference $((r_p - r_s)/c_o)$ which quantifies the difference in the propagation time between waves travelling to the field point from the primary and secondary sources respectively. When the field point is at a distance from the primary source which is greater than or equal to the distance from the secondary source (i.e., $r_p \geq r_s$) then we have the solution given by

$$h_o(\tau_2) = -\left(\frac{r_s}{r_p}\right)\delta\left(\tau_2 - \frac{r_p - r_s}{c_o}\right), \qquad r_p \geq r_s \qquad (15)$$

Thus the secondary source output is simply an inverted delayed version of the primary source output weighted by a factor accounting for spherical spreading. This result is not surprising. However, when the field point is further from the secondary source than the primary source, i.e., $r_s \geq r_p$ then it is clear that the secondary source must produce outputs prior to the primary source if cancellation of the field is to be achieved. Thus if $\eta = |r_p - r_s|/c_o$ we see that equation (14) can be written as

$$\left(\frac{r_s}{r_p}\right)R_{pp}(\tau_1 + \eta) + \int_0^\infty R_{pp}(\tau_1 - \tau_2)h_o(\tau_2)d\tau_2 = 0 \qquad \tau_1 > 0 \quad (16)$$

The problem has now reduced to the "pure prediction problem" of producing an optimal prediction of the output of a system at some time η in the future given the statistical properties of the signal. The solution to this type of problem is well-established (see reference [5] for full details) and can be easily written down provided that in this case the primary source strength time derivative $\dot{q}_p(t)$ can be expressed as the output of some linear "shaping" filter whose input is white noise. If the transfer function and impulse response of this filter are given by $Y_1(s)$ and $y_1(t)$ respectively, where s is the Laplace variable, then the transfer function of the optimal predictor whose impulse response satisfies equation (16) is given by

$$H_o(s) = -\left(\frac{r_s}{r_p}\right)Y_1^{-1}(s)Y_2(s) \tag{17}$$

In reference [6] full details are presented of the calculation of this transfer function when the shaping filter is of second order and has the form

$$Y_1(s) = \frac{s\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{18}$$

(A similar example has also been presented by Joseph *et al* [7].) The structure of the transfer function of the optimal filter that is produced is given by

$$H_o(s) = \left(\frac{r_s}{r_p}\right)(A/s + B) \tag{19}$$

where $A = (e^{-\zeta\omega_n\eta}/\omega_o)\omega_n^2\sin\omega_o\eta$, $B = (e^{-\zeta\omega_n\eta}/\omega_o)(\omega_o\cos\omega_o\eta - \zeta\omega_n\sin\omega_o\eta)$ and $\omega_o = \omega_n\surd(1 - \zeta^2)$.

Use of this result and equation (13) enables the expression for the minimum mean squared pressure to be written as

$$\frac{J_o}{J_{pp}} = 1 - \frac{e^{-2\zeta\omega_n\eta}}{\omega_o^2}\left[\omega_n^2\sin^2\omega_o\eta + (\omega_o\cos\omega_o\eta - \zeta\omega_n\sin\omega_o\eta)^2\right] \tag{20}$$

where J_{pp} defines the mean squared pressure due to the primary source only. A plot of J_o/J_{pp} as a function of ζ and the parameter $\omega_n\eta$ is shown in Figure 6. Note that the latter parameter can be written as $(2\pi|r_p - r_s|)/\lambda_n$ and thus quantifies the difference in path length between the primary and secondary sources to the field point considered relative to the acoustic wavelength λ_n at the natural frequency of the second order shaping filter. Thus, as one would expect, the greatest cancellation of the sound field is produced when the shaping filter is lightly damped and when the path length difference to the field point is small compared to the acoustic wavelength at the filter's natural frequency.

5. CONCLUSIONS

The work presented here formulates the problem of determining the optimal controller for stationary random sound fields when the constraint of causality is introduced. The problem reduces to a Wiener-Hopf integral equation that must be satisfied by the optimal vector of filter impulse responses. A simple example is presented for the single channel case which demonstrates that the degree of cancellation of a random sound field may in some circumstances be dependent on its statistical predictability. The extension of these techniques to determine how other problems such as the minimisation of acoustic power output [8] and acoustic potential energy [9] produces some further interesting results.

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APPENDIX

We wish to minimise the quadratic cost function consisting of the weighted sum of time averaged error signals, which for stationary ergodic random proceses can be written as

$$J = E[\underline{e}^T(t)\underline{\underline{W}}\underline{e}(t)] = \sum_{\ell=1}^{L} w_\ell E[d_\ell(t) - \hat{d}_\ell(t))^2] \qquad \text{(A1)}$$

where $\underline{W}$ is the diagonal matrix of weighting factors w_ℓ. Note that J can be written as

$$J = \sum_{\ell=1}^{L} w_\ell J_\ell \qquad \text{(A2)}$$

where

$$J_\ell = E[(d_\ell(t) - \hat{d}_\ell(t))^2]$$

Substituting equation (6) of the main text enables the ℓ'th contribution to the quadratic cost function to be written as

$$J_\ell = E[(d_\ell(t) - \int_0^\infty \underline{h}^T(\tau_1)\underline{r}_\ell(t-\tau_1)d\tau_1)^2] \qquad \text{(A3)}$$

which can be expanded to give

$$J_\ell = E[(d_\ell^2(t) - 2d_\ell(t)\int_0^\infty \underline{h}^T(\tau_1)\underline{r}_\ell(t-\tau_1)d\tau_1$$

$$+ \int_0^\infty\int_0^\infty \underline{h}^T(\tau_1)\underline{r}_\ell(t-\tau_1)\underline{r}_\ell^T(t-\tau_2)\underline{h}(\tau_2)d\tau_1 d\tau_2] \qquad \text{(A4)}$$

Note that use has been made of $\underline{h}^T(\tau_2)\underline{r}_\ell(t-\tau_2) = \underline{r}_\ell^T(t-\tau_2)\underline{h}(\tau_2)$ (since the term is a scalar) and τ_2 has been introduced as an additional time variable. Use of the expectation operator then gives

$$J_\ell = R_{dd\ell}(0) - 2\int_0^\infty \underline{h}^T(\tau_1)\underline{R}_{dr\ell}(\tau_1)d\tau_1$$

$$+ \int_0^\infty\int_0^\infty \underline{h}^T(\tau_1)\underline{R}_{rr\ell}(\tau_1-\tau_2)\underline{h}(\tau_2)d\tau_1 d\tau_2 \qquad \text{(A5)}$$

where the correlation functions are defined by

$$R_{dd\ell}(0) = E[d_\ell^2(t)], \qquad \underline{R}_{dr\ell}(\tau_1) = E[d_\ell(t)\underline{r}_\ell(t-\tau_1)],$$
$$\underline{R}_{rr\ell} = E[\underline{r}_\ell(t-\tau_1)\underline{r}_\ell^T(t-\tau_2)] \tag{A6}$$

Now we can add each of the L terms of this form to give the expression for the net cost function which can therefore be written as

$$J = R_{dd}(0) - 2\int_0^\infty \underline{h}^T(\tau_1)\underline{R}_{dr}(\tau_1)d\tau_1$$
$$+ \int_0^\infty\int_0^\infty \underline{h}^T(\tau_1)\underline{R}_{rr}(\tau_1-\tau_2)\underline{h}(\tau_2)d\tau_1 d\tau_2 \tag{A7}$$

where we have defined the weighted sums

$$R_{dd}(0) = \sum_{\ell=1}^{L} w_\ell R_{dd\ell}(0), \qquad \underline{R}_{dr}(\tau_1) = \sum_{\ell=1}^{L} w_\ell \underline{R}_{dr\ell}(\tau_1),$$

$$\underline{R}_{rr}(\tau_1-\tau_2) = \sum_{\ell=1}^{L} w_\ell \underline{R}_{rr\ell}(\tau_1-\tau_2) \tag{A8}$$

We can now derive the integral equation which must be satisfied by the optimal vector of filters $\underline{h}_o(\tau_1)$ which minimises the cost function J. The technique follows that presented in reference [4] which deals with the scalar case. We thus assume that

$$\underline{h}(\tau_1) = \underline{h}_o(\tau_1) + \epsilon\underline{h}_\epsilon(\tau_1) \tag{A9}$$

where ϵ is an arbitrary real parameter. This expression is substituted into equation (A7) to give

$$J = R_{dd}(0) - 2\int_0^\infty (\underline{h}_o^T(\tau_1) + \epsilon\underline{h}_\epsilon)^T\underline{R}_{dr}(\tau_1)d\tau_1$$
$$+ \int_0^\infty\int_0^\infty (\underline{h}_o^T(\tau_1) + \epsilon\underline{h}_\epsilon^T(\tau_1))\underline{R}_{rr}(\tau_1-\tau_2)(\underline{h}_o(\tau_2) + \epsilon\underline{h}_\epsilon(\tau_2))d\tau_1 d\tau_2 \tag{A10}$$

Expanding this expression and collecting terms in powers of ϵ gives

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$$J = R_{dd}(0) - 2\int_0^\infty \underline{h}_o^T(\tau_1)\underline{R}_{dr}(\tau_1)d\tau_1 + \int_0^\infty\int_0^\infty \underline{h}_o^T(\tau_1)\underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_o(\tau_2)d\tau_1 d\tau_2$$

$$+ \epsilon[-2\int_0^\infty \underline{h}_\epsilon^T(\tau_1)\underline{R}_{dr}(\tau_1)d\tau_1 + \int_0^\infty\int_0^\infty \underline{h}_o^T(\tau_1)\underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_\epsilon(\tau_2)$$

$$+ \underline{h}_\epsilon^T(\tau_1)\underline{R}_{rr}(\tau_1 - \tau_2)\cdot\underline{h}_o(\tau_2)d\tau_1 d\tau_2]$$

$$+ \epsilon^2[\int_0^\infty\int_0^\infty \underline{h}_\epsilon^T(\tau_1)\underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_\epsilon(\tau_2)d\tau_1 d\tau_2] \qquad \text{(A11)}$$

Now note that we can write the term

$$\underline{h}_o^T(\tau_1)\underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_\epsilon(\tau_2) = \underline{h}_\epsilon^T(\tau_2)\underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_o(\tau_1) \qquad \text{(A12)}$$

since it is a scalar (and equal to its transpose) and since the matrix $\underline{R}_{rr}(\tau_1 - \tau_2)$ is symmetric. Also, since $\underline{R}_{rr}(\tau_1 - \tau_2) = \underline{R}_{rr}(\tau_2 - \tau_1)$, then the integrated value of this term is symmetric to interchange of τ_1 and τ_2. This allows equation (A11) to be written as

$$J = R_{dd}(0) - 2\int_0^\infty \underline{h}_o^T(\tau_1)\underline{R}_{dr}(\tau_1)d\tau_1$$

$$+ \int_0^\infty\int_0^\infty \underline{h}_o^T(\tau_1)\underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_o(\tau_2)d\tau_1 d\tau_2$$

$$- 2\epsilon[\int_0^\infty \underline{h}_\epsilon^T(\tau_1)(\underline{R}_{dr}(\tau_1) - \int_0^\infty \underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_o(\tau_2)d\tau_2)d\tau_1]$$

$$+ \epsilon^2[\int_0^\infty\int_0^\infty \underline{h}_\epsilon^T(\tau_1)\underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_\epsilon(\tau_2)d\tau_1 d\tau_2] \qquad \text{(A13)}$$

This relationship can be expressed in the form

$$J = J_o + \Delta J \qquad \text{(A14)}$$

where J_o denotes the minimum value of J produced when $\underline{h}(\tau_1) = \underline{h}_o(\tau_1)$ and which is thus given by

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$$J_o = R_{dd}(0) - 2\int_0^\infty \underline{h}_o^T(\tau_1)\underline{R}_{dr}(\tau_1)d\tau_1$$

$$+ \int_0^\infty\int_0^\infty \underline{h}_o^T(\tau_1)\underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_o(\tau_2)d\tau_1 d\tau_2 \qquad \text{(A15)}$$

Now if $\underline{h}_o(\tau_1)$ is the optimal filter vector, then ΔJ must be greater than or equal to zero for all possible allowed values of $\underline{h}_\epsilon(\tau_1)$ and all values of $\epsilon \neq 0$. We thus need to establish the conditions under which $\Delta J \geq 0$. Firstly note that the second term constituting ΔJ (the last term of equation (A13)) can be written as

$$\sum_{\ell=1}^{L} w_\ell E[(\int_0^\infty \epsilon \underline{h}_\epsilon^T(\tau_1)\underline{r}_\ell(t - \tau_1)d\tau_1)^2] \qquad \text{(A16)}$$

and is thus the weighted sum of the expectations of a squared time history. This will certainly be greater than or equal to zero for all choices of ϵ and $\underline{h}_\epsilon(\tau_1)$ provided all the weighting factors w_ℓ are greater than or equal to zero. (It may also be possible for this to be greater than or equal to zero for other choices of weighting factors). However, the first term constituting ΔJ (the fourth term in equation (A13)) shows that there is a range of vlues of ϵ, which for a given $\underline{h}_\epsilon(\tau_1)$, can cause ΔJ to become negative. In particular $\Delta J < 0$ for

$$\epsilon < \frac{2\int_0^\infty \underline{h}_\epsilon^T(\tau_1)(\underline{R}_{dr}(\tau_1) - \int_0^\infty \underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_o(\tau_2)d\tau_2)d\tau_1}{\int_0^\infty\int_0^\infty \underline{h}_\epsilon^T(\tau_1)\underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_2(\tau_2)d\tau_1 d\tau_2} \qquad \text{(A17)}$$

Thus there is always a value of ϵ that will make ΔJ negative whether the numerator of this expression is positive or negative (since ϵ can be positive or negative). One therefore concludes that ΔJ can only be greater than or equal to zero for any values of ϵ and $\underline{h}_\epsilon(\tau_1)$ provided

$$\int_0^\infty \underline{h}_\epsilon^T(\tau_1)(\underline{R}_{dr}(\tau_1) - \int_0^\infty \underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_o(\tau_2)d\tau_2)d\tau_1 = 0 \qquad \text{(A18)}$$

A necessary and sufficient condition for this to hold for any value of $\underline{h}_\epsilon(\tau_1)$ is that

$$\underline{R}_{dr}(\tau_1) - \int_0^\infty \underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_o(\tau_2)d\tau_2 = 0 \qquad \tau_1 > 0 \qquad \text{(A19)}$$

This defines the necessary and sufficient condition for $\underline{h}_o(\tau_1)$ to constitute the optimal filter vector which minimises the chosen cost function. Finally note that this also enables a convenient relationship to be written for the minimum value of J. From equation (A15), this can be written as

$$J_o = R_{dd}(0) - \int_0^\infty \underline{h}_o^T(\tau_1)\underline{R}_{dr}(\tau_1)d\tau_1$$

$$-\int_0^\infty \underline{h}_o^T(\tau_1)(\underline{R}_{dr}(\tau_1) - \int_0^\infty \underline{R}_{rr}(\tau_1 - \tau_2)\underline{h}_o(\tau_2)d\tau_2)d\tau_1 \quad \text{(A20)}$$

and since $\underline{h}_o(\tau_1)$ must satisfy equation (A19) then

$$J_o = R_{dd}(0) - \int_0^\infty \underline{h}_o^T(\tau_1)\underline{R}_{dr}(\tau_1)d\tau_1 \quad \text{(A21)}$$

A further alternative means of writing this which is sometimes useful is given by recognising that

$$J_o = R_{dd}(0) - \int_0^\infty \underline{h}_o^T(\tau_1) \sum_{\ell=1}^{L} w_\ell E[d_\ell(t)\underline{r}_\ell(\tau - \tau_1)]d\tau_1 \quad \text{(A22)}$$

which can thus be written as

$$J_o = R_{dd}(0) - \sum_{\ell=1}^{L} w_\ell E[d_\ell(t)\int_0^\infty \underline{h}_o^T(\tau_1)\underline{r}_\ell(t - \tau_1)d\tau_1] \quad \text{(A23)}$$

The convolution term results in the optimal value of the signal $d_\ell(t)$ which can be written as $\hat{d}_{\ell o}(t)$ and therefore enables J_o to be written as

$$J_o = R_{dd}(0) - \sum_{\ell=1}^{L} w_\ell E[d_\ell(t)\hat{d}_{\ell o}(t)] \quad \text{(A24)}$$

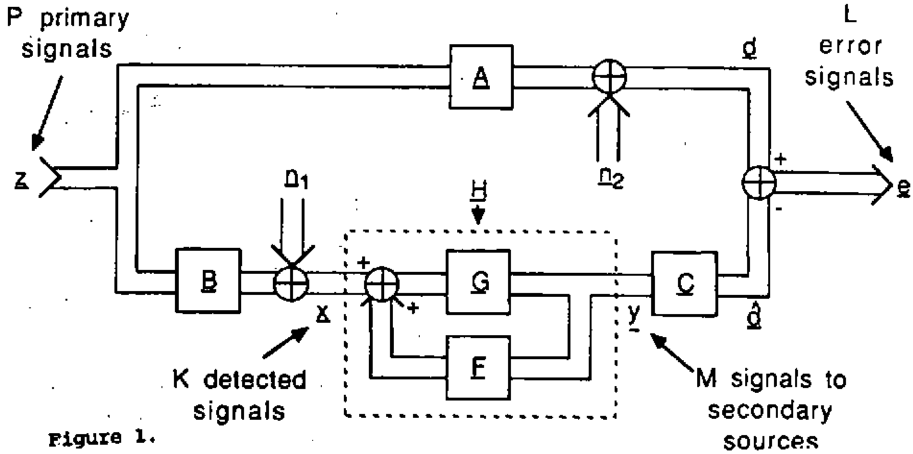

Figure 1.

The block diagram representation of the general feedforward active control problem. It is assumed that L ≥ M and K ≥ P. The signals $\underline{n}_1$ and $\underline{n}_2$ represent contaminating noise.

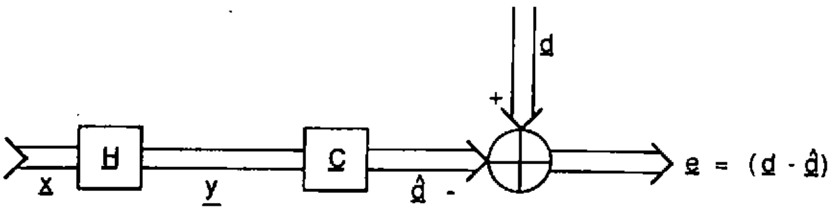

Figure 2. Summary of the optimal filtering problem. We wish to determine the optimal filter $\underline{H}_0$ which minimises $E[\underline{e}^T(t)\underline{W}\,\underline{e}(t)]$ where $\underline{W}$ is a diagonal matrix of weighting factors.

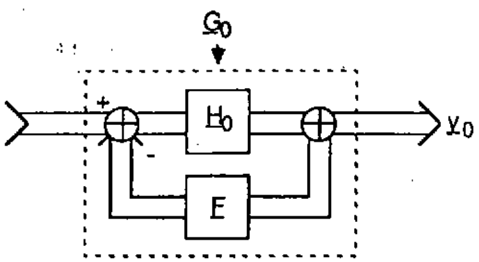

Figure 3. Block diagram representation of the optimal form of the filter $\underline{G}$ deduced in the frequency domain after determination of the optimal causal filter $\underline{H}_0$. Note that if $\underline{H}_0$ is causal, since $\underline{F}$ is causal, then $\underline{G}_0$ is causal (the stability of $\underline{G}_0$ is not guaranteed).

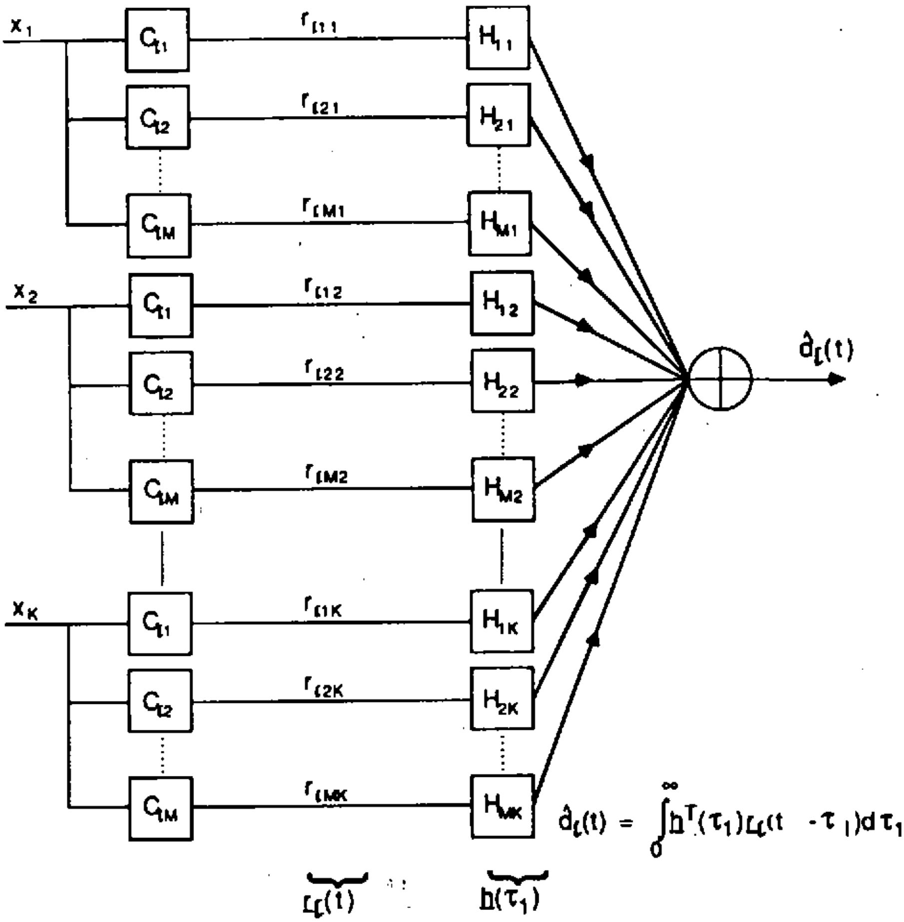

Figure 4. Equivalent block diagram showing the generation of the signal at the l'th error sensor by the secondary sources when the transfer functions $\underline{H}$ and $\underline{C}$ are reversed.

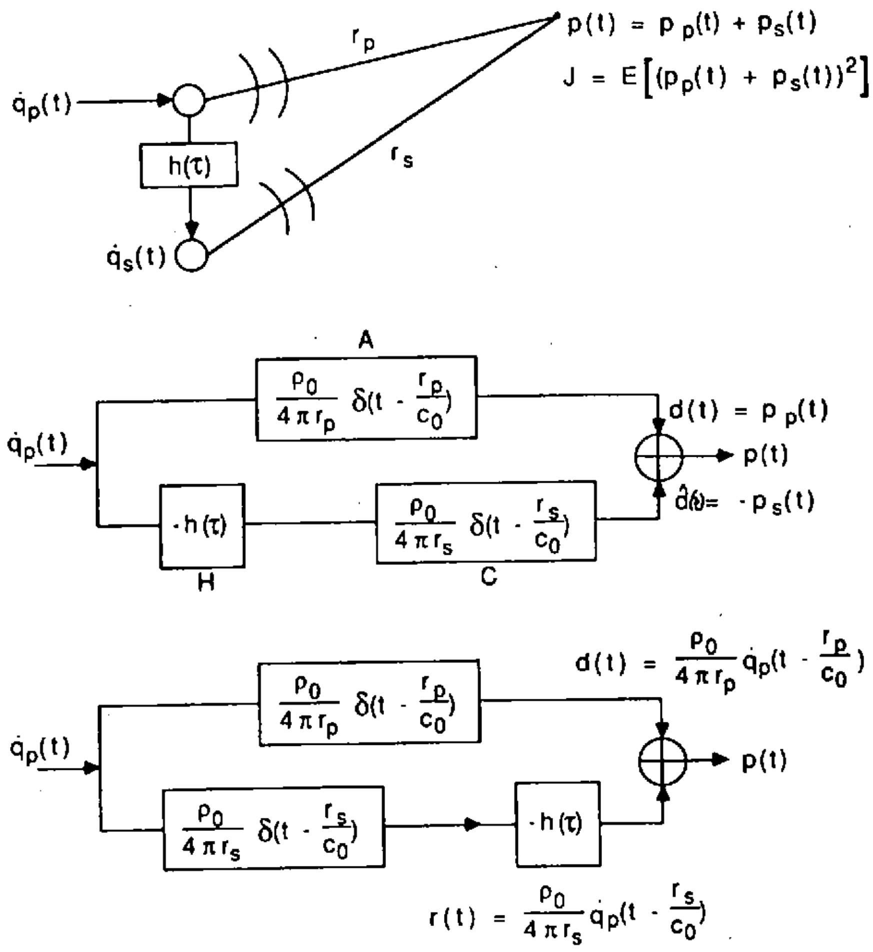

Figure 5. The minimisation of the far field acoustic pressure due to a single primary source with a single secondary source; (a) showing primary and secondary sources having source strength time derivatives $\dot{q}_p(t)$ and $\dot{q}_s(t)$ respectively, (b) the equivalent block diagram, (c) the block diagram with the transfer functions H and C reversed.

LINEAR LEAST SQUARES ESTIMATION PROBLEMS IN THE ACTIVE CONTROL OF STATIONARY RANDOM SOUND FIELDS

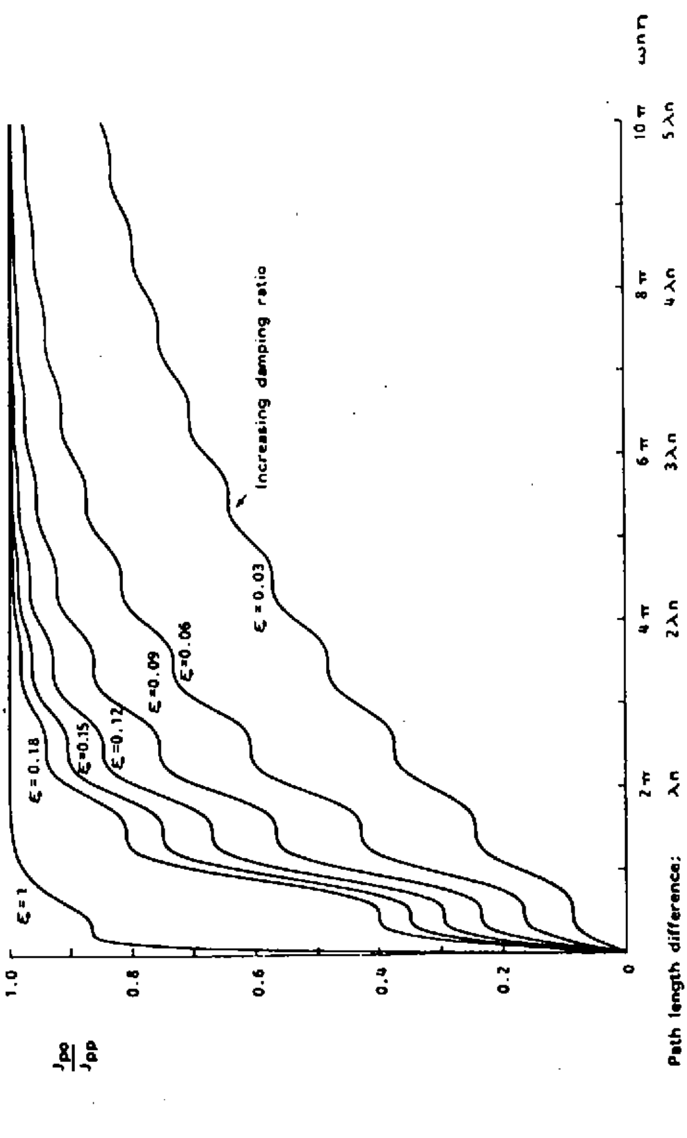

Figure 6. The normalised minimum mean squared pressure in the arc $r_p < r_B$ as a function of the damping ratio of the second order shaping filter and the parameter $\omega_n\eta$. The latter quantifies the difference in path length between the primary and secondary source and the field point relative to the acoustic wavelength at the natural frequency of the shaping filter.