ANALYSIS OF TRANSDUCERS USING PIEZOELECTRIC FINITE ELEMENTS AND ACOUSTIC BOUNDARY ELEMENTS

P. C. MACEY

PAFEC Limited, Strelley Hall, Nottingham NG8 6PE

1. INTRODUCTION

This paper considers methods of modelling piezoelectric transducers vibrating underwater. The finite element method is widely used for modelling vibration problems. However not many finite element programs can handle either piezoelectric materials or fluid-structure interaction. The present paper describes extensions to the PAFEC finite element program to incorporate both these features.

2. PIEZOELECTRIC FINITE ELEMENTS

Piezoelectric materials have a coupling between the mechanical and electrical properties; deforming a piece of piezoelectric material will cause a potential difference and conversely applying a potential difference will cause a deformation. Thus it is necessary to solve the elastic and electrostatic equations simultaneously.

The virtual work density is

$$W = \underline{u}^T \underline{F} - \phi q \qquad (1)$$

where $\underline{u}$ is the displacement, $\underline{F}$ the force, ϕ the potential and q the charge density. The constitutive equations are

$$\begin{aligned} \sigma &= c\,\epsilon - e\,E \\ D &= e^T \epsilon + \mathcal{E}\,E \end{aligned} \qquad (2)$$

where σ is the stress, ϵ the strain, E the electric field, D the flux density, c the elastic stiffness tensor and $\mathcal{E}$ the dielectric stiffness tensor. After applying the principle of virtual work, the finite element equations can be derived as in ref. [1]. These have the form

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$$\begin{bmatrix}[M_{uu}] & [0] \\ [0] & [0]\end{bmatrix} \begin{Bmatrix}\{\ddot{u}\} \\ \{\ddot{\phi}\}\end{Bmatrix} + \begin{bmatrix}[S_{uu}] & [S_{u\phi}] \\ [S_{\phi u}] & [S_{\phi\phi}]\end{bmatrix} \begin{Bmatrix}\{u\} \\ \{\phi\}\end{Bmatrix} = \begin{Bmatrix}\{F\} \\ -\{Q\}\end{Bmatrix} \qquad (3)$$

where {U} is a vector of displacements and $\{\phi\}$ is a vector of electric potentials.

There are analogies between the mechanical and electrical quantities. Displacement corresponds to electric potential, force to charge and strain to electric field. A connection to earth at a point can be modelled by setting the electric potential freedom to zero and removing the row and column from the stiffness and mass matrices and force vector. The effect of an electrode can be modelled by repeating the electric freedoms lying on this equipotential surface by collapsing the rows and columns in the stiffness and mass matrices and force vector.

The electric freedoms can be eliminated by static condensation on equation (3), giving

$$[M^*]\{\ddot{U}\} + [S^*]\{U\} = \{F^*\} \qquad (4)$$

where

$$\begin{aligned}[M^*] &= [M_{uu}] \\ [S^*] &= [S_{uu}] - [S_{u\phi}]\,[S_{\phi\phi}]^{-1}\,[S_{\phi u}] \\ \{F^*\} &= \{F\} + [S_{u\phi}]\,[S_{\phi\phi}]^{-1}\,\{Q\}\end{aligned} \qquad (5)$$

In order to perform this operation some electrical restraint must be applied to ensure that $[S_{\phi\phi}]$ is not singular. It is frequently advantageous to use Guyan reduction or static condensation to eliminate some mechanical degrees of freedom, reducing from large sparse matrices to small dense matrices, see ref. [2].

For harmonic vibrations at frequency ω, equation (4) simplifies to

$$([S^*] - \omega^2 [M^*])\,\{U\} = \{F^*\} \qquad (6)$$

Natural frequencies and mode shapes can be found by setting $\{F^*\}$ to zero and solving the eigenvalue problem.

For a transducer with 2 electrodes, solving the eigenvalue problem with one electrode earthed and the other electrode free (open circuit) will give the constant current drive resonances. Solving the eigenvalue problem with both electrodes earthed (short circuit) will give the constant voltage drive resonances, which will also be the antiresonances in the constant current drive case.

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4. FLUID EQUATIONS

We now consider the problem of determining the pressure distribution in an infinite region of compressible fluid surrounding a vibrating surface. The pressure P satisfies the wave equation

$$\nabla^2 p - \ddot{p}/c^2 = 0 \qquad (7)$$

where c is the acoustic wavespeed. For steady state harmonic problems, to which attention in this paper is confined, this reduces to the Helmholtz equation

$$\nabla^2 p + k^2 p = 0 \qquad (8)$$

where $k=\omega/c$ is the acoustic wavenumber, and ω is the frequency in radians/sec. The boundary condition

$$P_{,n} = -i\omega\rho V \qquad (9)$$

, where ρ is the density, enforces continuity of normal velocity V on the vibrating surface. Finally the equation

$$\lim_{r\to\infty} r|P_{,r} - ikP| = 0 \qquad (10)$$

is needed to ensure that the pressure field consists only of outgoing waves.

5. SURFACE HELMHOLTZ FORMULATION

By using the divergence theorem it can be shown that the pressure at a point $\underline{x}$ satisfies the equation

$$\epsilon p(\underline{x}) = \int_{\Gamma} (p(\underline{y}) g_{,n(\underline{y})}(\underline{x},\underline{y}) - g(\underline{x},\underline{y}) p_{,n(\underline{y})}) d\Gamma(\underline{y}) \qquad (11)$$

where

$$g(\underline{x},\underline{y}) = \exp(-ikr)/4\pi r \qquad (12)$$

with $r=|\underline{x}-\underline{y}|$, is the free space Green's function and $4\pi\epsilon$ is the solid angle in the exterior region. To obtain numerical solutions the surface Γ can be divided into patches over which the pressure P and its normal derivative $P_{,n}$ are interpolated using the same shape functions [N]. Thus equation (11) becomes

$$\epsilon P(\underline{x}) - \sum_{i=1}^{n} \int_{P_i} \frac{\partial g}{\partial n_y} [N]\, d\Gamma\, \{P\} = -\sum_{i=1}^{n} \int_{P_i} g\, [N]\, d\Gamma\, \{P_{,n}\} \qquad (13)$$

where P_i is the ith patch on the boundary element and n is the number of patches. Taking $\underline{x}$ to be at each nodal point in turn, a set of linear equations are obtained which can be written in matrix form as

$$[H]\ \{P\} = [G]\ \{P_{,n}\} \qquad (14)$$

There are some difficulties in evaluating some of the terms in equation (13). When the patch P_i contains the collocation point $\underline{x}$ the integrand becomes singular. The integral still converges to a finite result, but Gaussian integration is unsuitable for evaluating it. The integral can be evaluated by a variety of methods. In the PAFEC system the method of Lachat and Watson [4] is employed. The patch is decomposed into triangles with the collocation point at the apex. A unit square is transformed onto the triangle such that one side collapses to the collocation point. The zero in the jacobian cancels with the singularity in the original integrand, giving a bounded function, when integration is performed over the unit square. In the PAFEC acoustics system the 3D boundary elements are constructed from generally curved 8-noded quadrilateral and 6-noded triangular patches. The shape functions [N] can be chosen either the same as the quadratic functions used to interpolate the geometry, or to be constant. In the latter case the collocation points are taken to be at the centroid of each patch. There is also an axisymmetric acoustic boundary element constructed from 3-noded quadratic line patches.

If the surface normal velocities and hence normal pressure gradients are known, then by equation (14) the surface pressures can be determined, and by using equation (11) for $\underline{x}$ in the fluid the surrounding pressure distribution can be found. Unfortunately equation (14), the surface Helmholtz formulation, fails at certain characteristic frequencies. This can be seen from fig. 5 where the ratio of pressure over its normal derivative for a pulsating spherical surface has been computed using the mesh shown in fig. 6. A modified Green's function was used to take account of the remaining parts of the spherical surface not modelled. The numerical solution gives good results except at certain frequencies where it goes quite wrong. The reason for this failure can be understood by considering the interior problem. Internal regions can also be modelled by the boundary element method; an equation similar to (14) is obtained except that [H] is opposite in sign and also modified on the diagonal. However the interior acoustic problem with the Dirichlet boundary condition has unbounded resonances, and thus at these frequencies [G], common to both internal and external problems, must be singular. As the external problem does not have an unbounded resonance [H] must also be singular at the same frequency. At nearby frequencies the matrices are ill-conditioned and inaccurate results are obtained. The problem lies not in the original radiation problem, equations (8), (9) and (10), but in the integral equation (11).

6. MORE SOPHISTICATED BOUNDARY ELEMENT METHODS

The failure of the surface Helmholtz formulation occurs because equation (11), with $\underline{x}$ restricted to the surface, does not have a unique solution at the critical frequencies. Two boundary element methods, CHIEF and CONDOR, are commonly used to overcome this problem.

CHIEF, combined Helmholtz integral equation formulation, was first introduced by Schenck [5]. The equations of the surface Helmholtz formulation are augmented by some additional equations, taking $\underline{x}$ at some interior points in equation (11), when $\epsilon = 0$. The resulting overdetermined system is solved by least squares methods. This set of equations produced has a unique solution provided that not all the interior collocation points lie at nodes of the interior eigenfunctions. At higher frequencies these interior nodes become more numerous and the difficulty is knowing how many interior points to select and where to position them. In the PAFEC system the CHIEF method can be used with all the acoustic boundary elements.

CONDOR, composite outward normal derivative overlap relation, was first introduced by Burton and Miller [6]. This method combines the surface Helmholtz formulation, and its normal derivative

$$\epsilon\, p_{,n}(\underline{x}) = \int_{\Gamma} \left(p(\underline{y})\, g_{,n(\underline{y}),n(\underline{x})} - g_{,n(\underline{x})}\, p_{,n(\underline{y})} \right) d\Gamma(\underline{y}) \qquad (15)$$

It can be shown that if a complex combination of these two equations is taken, then the resulting formulation does not suffer from any characteristic frequencies. The difficulty in the CONDOR method is evaluating the highly singular integrals. Burton and Miller [6] suggest a regularization method expressing the highly singular operator as the product of two weakly singular operators. This leads to a double surface integration and requires much computation. In the PAFEC acoustic system a method suggested by Meyer et al. [7] is employed. This is based on the relation

$$\int_{\Gamma} P(\underline{y})\, g_{,n(\underline{x})n(\underline{y})}\, d\Gamma = \int_{\Gamma} (P(\underline{y}) - P(\underline{x}))\, g_{,n(\underline{x})n(\underline{y})}\, d\Gamma - P(\underline{x}) \int_{\Gamma} \underline{n}_x \cdot \underline{n}_y\, (ik)^2\, g\, d\Gamma \qquad (16)$$

When the boundary element is composed of constant pressure patches, the highly singular part vanishes when integrating over the collocation point patch.

7. EXAMPLE RADIATION PROBLEM

The problem of determining the far field radiation pattern for a cylindrical curved surface vibrating in an infinite external fluid was considered. The cylinder was taken to be of radius 1m and length 4m, see fig. 7. The curved surface was assumed to vibrate with unit normal velocity and the flat ends to remain stationary. The surrounding fluid was taken to be water. An alternative numerical solution is available, due to Williams et al. [8]. The mesh used to analyse this problem consisted of 12 constant pressure patches, modelling half axially and 30 degrees circumferentially, see fig. 8. The CONDOR formulation was used. Figures 9, 10 and 11 show the normalized far field radiation patterns for k=1, k=2 and k=5. There is agreement with the alternative solution.

8. COUPLED FLUID-STRUCTURE PROBLEMS

To couple structural finite elements and an acoustic boundary element some coupling matrices [E] and [T] must be evaluated, where $[E]^T\{U\}$ gives the set of normal displacements on the fluid mesh, and $-[T]^T\{P\}$ gives the forces on the structural mesh. Details of the construction of these matrices is given in ref. [9].

The coupled set of equations can now be written in the form

$$\begin{bmatrix} [Q_{ss}] & [Q_{sb}] \\ [Q_{bs}] & [Q_{bb}] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{P\} \end{Bmatrix} = \begin{Bmatrix} \{g_s\} \\ \{g_b\} \end{Bmatrix} \qquad (17)$$

where

$$[Q_{ss}] = [S] - \omega^2 [M]$$

$$[Q_{sb}] = [T]^T$$

$$[Q_{bs}] = -\omega^2 \rho [G] [E]^T$$

$$[Q_{bb}] = [H]$$

$$\{g_s\} = \{F\} \qquad (18)$$

$$\{g_b\} = \{0\}$$

Equation (17) can be solved for the structural displacements and the pressures on the fluid-structure interface. These results can then be used with equation (11) to determine the pressures within the fluid.

9. TRANSDUCER VIBRATING IN WATER

The method outlined in the previous section was used to analyse the ring transducer of fig. 1 vibrating in water. For the structure a mesh of twice the mesh density of that shown in fig. 2 was used and an acoustic boundary element of 40 constant pressure patches, matching face to face with the structural mesh, was used to model the fluid. The CONDOR method was employed. Figs 12 and 13 show the pressure distribution 1m away from the transducer for the two frequencies of 3000 Hz and 7000 Hz. Agreement is good except in the axial direction.

10. CONCLUSIONS

It has been shown that the finite element method can be used to model vibration of piezoelectric materials and the boundary element method can be used to solve radiation problems. These techniques have been successfully incorporated into the PAFEC finite element program. Comparison between F.E. and experimental results is good except in the axial direction, where there is as yet an unresolved discrepancy.

11. REFERENCES

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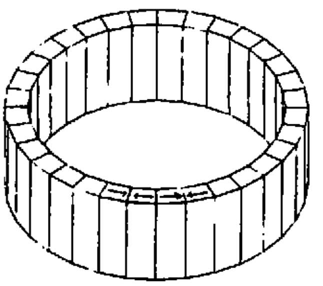
CYLINDER TRANSDUCER

Fig 1

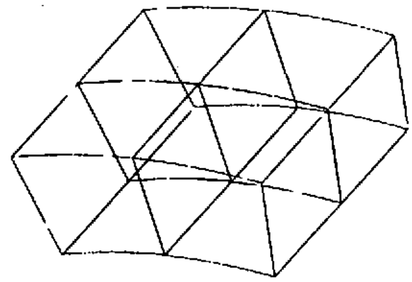


staves are tangentially polarized in alternate directions (arrows)

The staves are separated by electrodes, alternatively anodes and cathodes

outer radius : 0.1937m
inner radius : 0.1645m
height 0.127m

Fig 2



mesh used to model [illegible]

Fig 3

Comparison of experimental and F.E. impedances for tangentially polarized cylinder transducer vibrating in vacuo (Ohms)

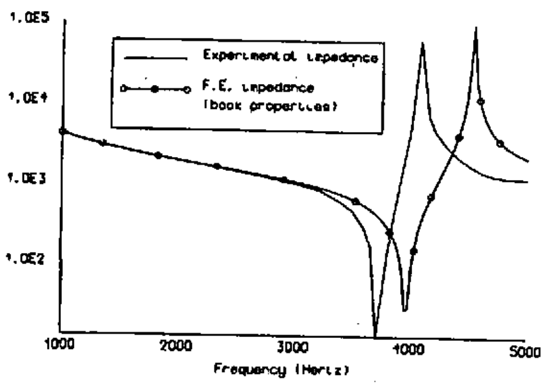


Fig 4

Comparison of experimental and F.E. impedances for tangentially polarized cylinder transducer vibrating in vacuo (Ohms)

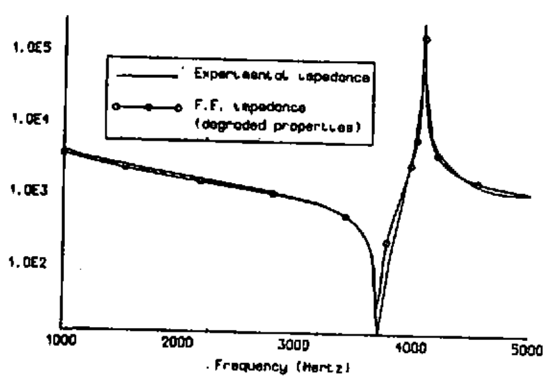


Fig 5

Ratio of pressure over radial derivative for a uniformly pulsating spherical surface comparison of analytical and surface Helmholtz integral formulation results

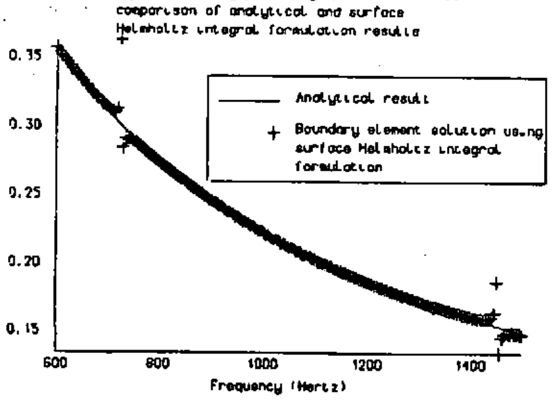
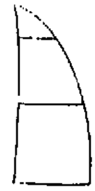
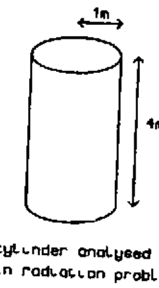


Fig 6



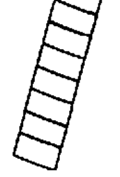
acoustic boundary element used to analyse pulsating spherical surface

Fig 7



cylinder analysed in radiation problem

Fig 8



acoustic boundary element used to analyse radiating cylindrical surface

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Fig 9

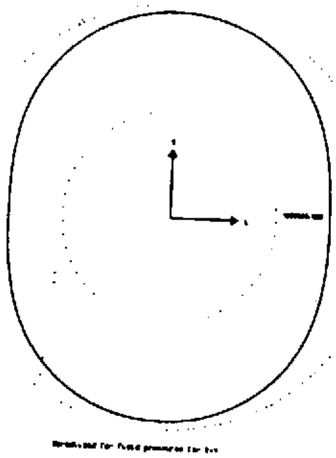

Fig 10

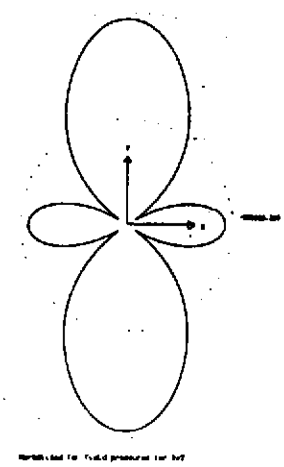

Fig 11

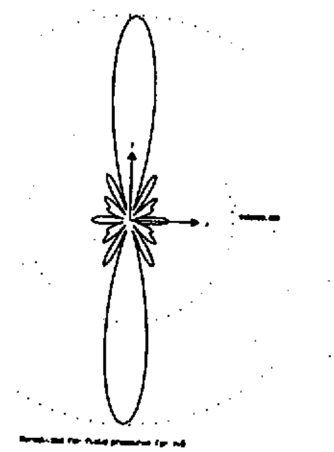

Fig 12

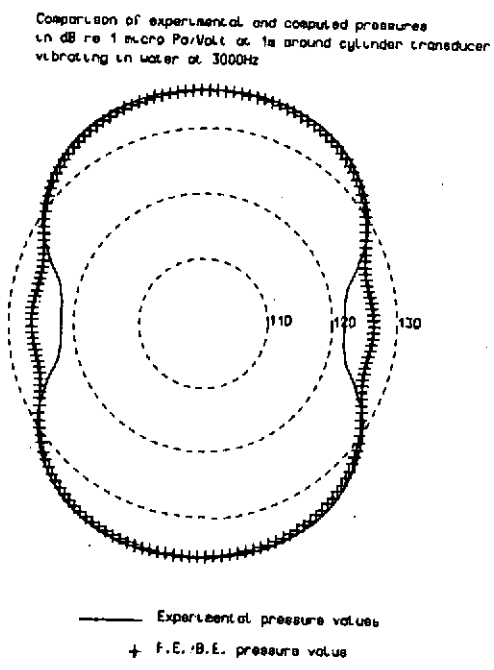

Fig 13