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SOURCES OF RADIATED ACOUSTICAL WAVES IN FLUIDS

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1. INTRODUCTION, BASIC EQUATIONS AND THE TIME-STATIONARY ASSUMPTION

This paper is both similar to and complementary to a paper (entitled "The physics of aeroacoustics") to be presented at Noise '93, St Petersburg, Russia, 31 May - 3 June 1993 [1]. In that paper a case is made out for regarding the fluctuating total enthalpy field $H'(x_k,t)$ as an important and potentially useful dependent "acoustic" field. Due to space restrictions, derivations of the two principal equations are not given in that paper. The first of these is for the divergence of the "total enthalpy mean radiating acoustic intensity", $\overline{H'm'_i}$, where the overbar indicates a suitable time average, the primes denote purely fluctuating quantities, and $m_i \equiv \rho v_i$ is the linear momentum density. The second is an inhomogeneous "wave equation" for H'. Outline derivations of these two equations are given here in the Appendix, and the text provides some additional discussion of what can be learned from the equations about the physical mechanisms that can produce radiated acoustic waves in fluids. (Unavoidably, there is considerable duplication in the two papers.)

Standard forms of the transport equations of mass, linear momentum and energy, respectively, of a Stokesian fluid are

$$\partial\rho/\partial t + \partial(\rho v_i)/\partial x_i = 0, \quad \partial(\rho v_i)/\partial t + \partial p/\partial x_i + \partial(\rho v_i v_j - S_{ij})/\partial x_j = \rho f_i, \qquad (1,2)$$

$$\partial(\rho U + \tfrac{1}{2}\rho v_i^2)/\partial t + \partial[(\rho U + \tfrac{1}{2}\rho v_i^2)v_j + p v_j - S_{ij}v_i - \lambda\partial T/\partial x_j]/\partial x_j = \rho v_i f_i + \rho Q. \qquad (3)$$

The notation is standard, except that Q is the rate of external heat addition per unit volume and

$$S_{ij} = \mu[\partial v_i/\partial x_j + \partial v_j/\partial x_i - 2(\partial v_k/\partial x_k)\delta_{ij}] + (\tfrac{4}{3}\mu + \eta)(\partial v_k/\partial x_k)\delta_{ij} \qquad (4)$$

is the Stokesian viscous stress tensor, in which η is the coefficient of bulk viscosity; also the "external" force per unit mass f_i is understood to include forcing due to chemical reactions (e.g., combustion or absorption/radiation of electromagnetic energy) and gravitation. Together with these transport equations and their implied Stokesian and Fourier constitutive equations for the stress tensor, $p_{ij} = p\delta_{ij} - S_{ij}$, and the heat flux, $= -\lambda\partial T/\partial x_i$, respectively, one has the following thermodynamic relationships for each fluid mass element:

$$p = R\rho T, \quad \delta U = c_v\delta T = T\delta S - p\delta(1/\rho), \quad \delta h = c_p\delta T = T\delta S + (1/\rho)\delta p. \qquad (5a,b,c)$$

Here p is the thermodynamic pressure (as it is in the stress tensor), R is the fluid's ideal gas constant, c_v and c_p are the specific heats at constant specific volume ($V = 1/\rho$) and constant pressure (p), respectively, T is the temperature, S is the entropy and h is the enthalpy. One also has the definitions

$$\gamma = c_p/c_v, \quad c^2 = \gamma p/\rho, \quad H = h + \tfrac{1}{2}v_i^2. \qquad (6a,b,c)$$

H is often called the "stagnation enthalpy" but here the term "total enthalpy" is preferred.. Apart from the gas constant R none of the quantities in the differential system of equations (1), (2), (3) and (5) and in the definitions (4) and (6) are assumed to be constants in the analysis which follows; any or all of the others may be functions of position x_k and time t. The only assumption is time-stationarity of the flow. This means, first, that any quantity, $p(x_k,t)$ for example, is the sum of a unique time-averaged part and a unique purely fluctuating part of zero time average. Thus, in the case of $p(x_k,t)$, one has

$$p(x_k,t) = \bar{p}(x_k) + p'(x_k,t), \quad \bar{p}(x_k) = \int_{t_1}^{t_2} p(x_k,t)\,dt/(t_2 - t_1), \quad \int_{t_1}^{t_2} p'(x_k,t)\,dt/(t_2 - t_1) = 0, \qquad (7)$$

and similarly for all other quantities. This part of the assumption is not an assumption at all, but simply a definition of mean and fluctuating parts for any given time interval (t_1,t_2). Physically, whatever measurements one makes of however many field quantities at however many spatial positions during the same time interval (t_1, t_2), the records of each of these can be separated into such unique parts by performing the time averaging process shown in the second of equations (7) for each record; this gives $\bar{p}$ and p' follows as $p - \bar{p}$. The second part of the assumption is restrictive, however. This is that the time derivative of the fluctuating part of any quantity has zero time average. In the case of $p'(x_k,t)$, for example, the time average of $\partial p'/\partial t$ is $[p'(x_k,t_2) - p'(x_k,t_1)]/(t_2 - t_1)$, and this is zero only if (i) the fluctuations are periodic and $(t_2 - t_1)$ is an integer multiple of the period, or (ii) $p'(x_k,t_2)$ and $p'(x_k,t_1)$ are both zero, which would be the case if the fluctuating motion were of finite duration (i.e., a transient), non-zero only for $t_1 < t < t_2$, or (iii) the motion is of a random character and $t_2 - t_1$ is long enough. Each of these three possibilities can be a reasonable approximation to reality in situations of practical interest.

The idealized physical problem of interest here is essentially the same as Lighthill's well known acoustic analogy problem, except that the real fluid is not replaced by a hypothetical acoustic medium. One has a disturbed flow "source region", of effectively finite extent, V_S, surrounded by an infinite extent of the fluid which is effectively in a state of uniform static equilibrium apart from small amplitude fluctuations, acoustic waves of p', say, with corresponding ρ', T' and irrotational velocity fluctuations. These two regions are "effectively" distinct, as described, but the reality is that the strength of the sources tends to weaken sufficiently with distance, so that whatever acoustic disturbances there may be become dominant. The only essential hypothesis is that the fluid disturbances satisfy the Sommerfield radiation condition as the distance tends to infinity. This rather idealized picture has basic physical validity for real, weakly viscous and thermally conducting fluids, as it is the adiabatic acoustic waves which can travel out much further from a generally disturbed flow region than the vortical and entropic fluctuations which can "propagate" only by means of convection and slow diffusion (even shock waves such as "sonic booms" turn acoustic before dying away altogether!). Watching the vapour trails and listening to the noise of a jet airliner as it travels past overhead is an "experiment" tending to confirm this physical validity. Finally, as to the model, it must be mentioned that inside V_S, but not part of it, is the surface S_E of the structure of the machine producing the disturbed flow (see Figure 1).

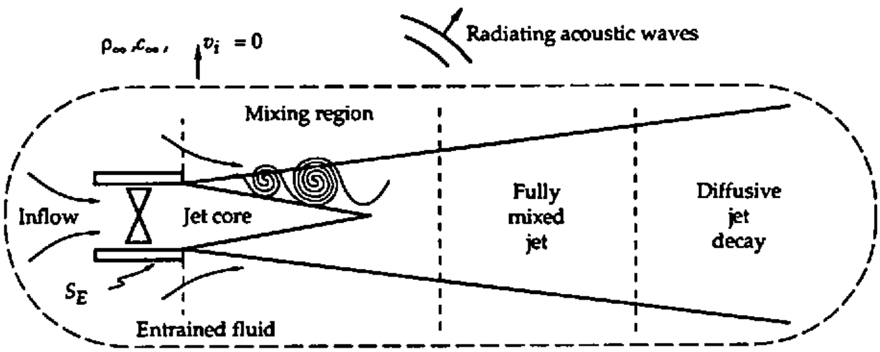
2. THE TWO EQUATIONS INVOLVING H'

2.1 The Radiated Intensity Equation

The first equation which is applicable to this idealized physical problem, and which provides an identification of the sources of the time-averaged radiated acoustic intensity in the far field, can be obtained by using the same methods as in reference [2] (the earliest version of this expression was presented by the author in a verbally delivered paper at the 1974 Eighth International Congress on Acoustics in London). The expression is derived from equations (2) and (3), with use of the time-stationary assumption, as outlined in the Appendix, and is

$$\partial(\overline{H'm'_i})/\partial x_i = \overline{m'_i[f'_i - (\omega' \times \overline{m})_i + (T\partial S/\partial x_i)' + (\rho^{-1}\partial S_{ij}/\partial x_j)']} + (\overline{p'\partial S'/\partial t})/R. \qquad (8)$$

Here ω is the Beltrami vorticity, Ω/ρ where $\Omega = \nabla \times \mathbf{v}$ is the vorticity. As was shown in reference [2], the mean total enthalpy intensity vector $\overline{H'm'_i}$ (J_i for brevity) reduces to the usual acoustic mean intensity vector $\overline{p'v'_i}$ in the case of small amplitude adiabatic fluctuations in an otherwise static and uniform fluid. By the same methods as used in reference [2] (see section 5.2), one can prove, for our problem as stated in the preceding section, that the mean total radiated acoustic power, W, is given by



Effective boundary of disturbed flow region V_S

Figure 1 Schematic of a representative problem (not to scale)

$$W = \int_{V_S} q_J(\mathbf{x})\, dV_S(\mathbf{x}) + \int_{S_E} J_i(\mathbf{x})\, dS_{Ei}(\mathbf{x}), \tag{9}$$

where q_J is the complete source term given by the right side of equation (8), J_i includes its solenoidal part, and the direction of $dS_{Ei}(\mathbf{x})$ is into V_S. This result is easily obtained by expressing the Green formula solution of the Poisson equation for the scalar potential of J_i in the well known spherical harmonics expansion form and noting then that only its zero order term, proportional to $1/r$, where r is the distance of the observer from the origin (which is somewhere convenient in the engine, say), gives the part of J_i proportional to $1/r^2$, and hence the acoustic power output as r tends to infinity. Rapid fluctuations in the thermodynamic properties of a fluid particle are known to be nearly adiabatic in weakly viscous and thermally conducting fluids; since entropy can enter or leave the particle only by slow diffusion, the particle's entropy is slowly varying. Also, the action of viscosity is via a similarly slow diffusion process, and so the vorticity of a fluid particle is similarly slowly varying. Accordingly, in many disturbed flows of interest, the dependence of q_J on its entropy and viscous stress terms (see equation (8)) is likely to be appreciably weaker than its dependence on the Coriolis acceleration term. Hence, to zero order in viscous and entropic effects, q_J reduces to simply $\overline{m_i\,(\omega' \times \mathbf{m}')_i} - \overline{m'_i f'_i}$). The external force f'_i is zero, of course, in the absence of gravitational, electromagnetic, and chemical reaction (combustion) forces. S_E is the surface only of the engine structure, so the q_J source region V_S includes that part of the engine occupied by the fluid. The contribution to W of the S_E integral in equation (9) depends only on the component of J_i normal to the surface. Real engines have moving parts, and the engine housing may be in motion. Both the volume and surface integrals in equation (9) are expressed in the observer's fixed coordinate system, and in this system both V_S and S_E in practice will be functions of time. Accordingly, to deal with both engine housing motion and engine parts motion relative to it, the time averaging must account for this V_S and S_E time dependence. The integrals then become appreciably more complicated looking mathematically, but retain essentially simple physical interpretations. The normal component of J_i remains the normal component when S_E is in motion.

2.2 The Inhomogeneous Fluctuating Total Enthalpy "Wave Equation"

Howe [3], in 1975, first called attention in published work to the relevance of the total enthalpy and the Coriolis acceleration to the theory of sound generated aerodynamically. The author had been independently developing rather similar ideas, beginning in about 1973, which were reported only verbally in his ICA paper of 1974, as mentioned in section 2.1; subsequent progress was as described in the Preface of reference [2]. Although H is directly a function of only the temperature and the square of the velocity (H/c_p is called the total temperature in fluid mechanics), for adiabatic fluctuations one can equally well regard the pressure p' as the fluctuating thermodynamic quantity upon which H' depends. To first order in p' one has $H' = (p'/\bar{p}) + \frac{1}{2}(v^2{}_i)'$. Hence $\bar{\rho}H'$ is a fluctuating "total pressure", which is zero if the total pressure concerned is of Bernoulli type: i.e., for the first order example, if $p' + \frac{1}{2}\bar{\rho}(v^2{}_i)'$ is constant. In aeroacoustics this has the advantage that H' is explicitly independent of such convected pressure fluctuations. Lord Rayleigh chose the fluctuating temperature as the dependent variable in his discussion of the effects of viscosity and thermal conductivity on sound waves [4]. That problem has something in common with the aeroacoustics problem: in Rayleigh's problem the equations for the pressure and entropy fluctuations are inextricably coupled; so are they in the aeroacoustics problem, and in addition the equation for the vorticity fluctuations is coupled to both the others, so that there are no independent acoustic, vortical and entropic "modes of motion".

For these reasons it is desirable to have a generally valid equation for H', of a form which may provide some physical insights. Such an equation can be derived, as outlined in the Appendix. It is

$$\partial^2 H'/\partial x_i^2 - [c^{-2}\{\partial^2 H'/\partial t^2 + 2\{v_i\partial/\partial t + \tfrac{1}{2}(\Omega\times v)_i + \tfrac{1}{2}V_i - \partial h/\partial x_i\}\,\partial H'/\partial x_i + v_i v_j\,\partial^2 H'/\partial x_i\,\partial x_j\}]'$$
$$= -\partial[(\Omega\times v)'_i - V'_i]/\partial x_i - \partial[R^{-1}DS/Dt]'/\partial t + [(\partial c^{-2}/\partial t)\,Dh/Dt]' - [(v_i/c^2)\partial V'_i/\partial t]'$$
$$+ 2[c^{-2}\{\tfrac{1}{2}v_i v_j\,\partial/\partial x_j + \tfrac{1}{2}(\Omega\times v)_i + \tfrac{1}{2}V_i - \partial h/\partial x_i\}\,\{(\Omega\times v)'_i - V'_i\}]' . \qquad (10)$$
$$V_i = T\partial S/\partial x_i + (1/\rho)\partial S_{ij}/\partial x_j + f_i \qquad (11)$$

is used for notational brevity in equation (10) to denote the combined entropic, viscous and external force terms. The form of equation (10) is admittedly one which has been contrived, but this has been done with the aim of producing terms both as concise as possible and in groups representing physical aspects of the motion which are reasonably distinct and different. The primes denote "the fluctuating part of". They arise in the derivation whenever it is only the fluctuating part of a single quantity which is being considered or the group of terms thus labelled has arisen from a time differentiation of a larger group. Individual terms and groups of terms without a prime have both mean and fluctuating parts. The aim has been achieved inasmuch as the terms in the equation do have distinct physical interpretations. First, with one exception, the enthalpy h and the squared sound speed c^2, and their derivatives, can be regarded as simply temperature dependent coefficient factors for their respective terms; h and c^2 of course are usually the same thing, since, for example, for a fluid with constant specific heats one has $c^2 = (\gamma - 1)h$. The exception is the term $[(\partial c^{-2}/\partial t)Dh/Dt]'$ on the right side of the equation, which is regarded as a source distribution for H' due to fluctuations in the speed of sound. The left side of equation (10) is evidently of convected wave equation form, but with the usual $v_i\partial/\partial x_i$ operating on $\partial H'/\partial x_i$ augmented by the term $\frac{1}{2}(\Omega\times v)_i + \frac{1}{2}V_i$ and a term due to spatial and temporal variation in the speed of sound, $-\partial h/\partial x_i$.

It is surprising (to the author at least) that the source terms of equation (10) form such relatively simple and distinct groups. The first and last groups are the effects of the combined Coriolis, entropic, viscous stress and external fluctuating forces, given by $\{(\Omega \times \mathbf{v})'_i - V'_i\}$ (note that $(\Omega \times \mathbf{v})_i$ is equal to $(\omega \times \mathbf{m})_i$, which was used in section 2.1) The second term involves only the entropy and its convection, and the third term, similarly, only the speed of sound (temperature) and its convection. Note that, since $-m'_i(\Omega \times \mathbf{v})'_i$ is equal to $\bar{m}_i(\omega' \times \mathbf{m}')_i$, the first four terms of q_I as shown in equation (8) are equal to the time average of $-m'_i\{(\Omega \times \mathbf{v})'_i - V'_i\}$. Perhaps equally surprising is the linearity in H' of equation (10). Taken together, these two surprising features suggest that although the equation has been mathematically contrived it may well be physically sensible. In this context it does have one unusual aspect, however. The only irreversible process terms appearing on the left side of the equation are those in the multiplicative coefficient $\frac{1}{2} V_i$ of $\partial H'/\partial x_i$ and clearly these will not give rise to the usual attenuation of small amplitude acoustic waves by viscous and thermal diffusion. This attenuation, therefore, must be largely provided by the source terms depending on the fluctuating viscous stresses and entropy. Although this representation of attenuation as due to "source terms" is not usual in wave motion it is physically acceptable. Attenuating diffusion effects are those of sinks of the otherwise conserved wave energy, and can be legitimately regarded as such, especially in the present problem. As Rayleigh showed in his problem, weak diffusive and adiabatic wave motions can be treated as uncoupled to zero order in the Stokes number, and the effect of either one upon the other can be introduced to first order as a perturbation of the zero order motion.

This linearity of equation (10) in H' has the important physical consequence that if all the source terms are zero in a region of the fluid then any H' in that region must have been generated elsewhere. This is rigorously true on mathematical grounds since one then has a linear homogeneous equation which, by itself, has no unique solution. Even when the coefficients are functions of position and time, there can be no "parametrically driven" oscillations of H' without some external forcing, or forcing at the region's boundary.

3. CONCLUSIONS AND DISCUSSION

It has been demonstrated that the fluctuating total enthalpy H' is a physically desirable and mathematically convenient dependent variable for all aeroacoustic problems, including both subsonic and supersonic flows and flows with combustion. This generality is claimed because both heat addition and forcing due to combustion have been taken into account, and the equations obtained are not subject to any restrictions on flow speeds. The time-stationarity assumption is not unduly restrictive for fluctuating flows of interest in aeroacoustics. An inhomogeneous convected wave equation for H' has been presented. Its two most important features, due to the fact that H' is independent of Bernoulli-type pressures, are as follows: (i) it is linear in H', and the coefficients of the H' terms do not implicitly contain H' as such, being comprised of factors each of which is a function of velocity or temperature, but not in a combination forming H'; the homogeneous terms are therefore "properly linear" in H', and cannot be confused with source terms; this linearity is preserved whatever dependence the coefficients may have on position and time; (ii) its source terms are similarly independent of H', and are physically distinct; the terms respectively represent, mostly separately, the effects of fluctuations in Coriolis acceleration, entropy, viscous stresses, temperature and external forces; for a lossless fluid, for example, under no external forces and subject to no external heat addition, the only non-zero source terms are those dependent on the fluctuations in the Coriolis acceleration, in the temperature times the entropy gradient, and in the speed of sound (temperature).

An expression has been presented for the divergence, q_J, of the mean total enthalpy intensity $J_i = \overline{H'm'_i}$, where m_i is the linear momentum density. By using this, a simple expression for the radiated acoustic power from a disturbed flow region can be found, and this has been presented. The solenoidal component of J_i has not been discussed here, because it does not contribute to power output of volume sources. It is needed to determine the total mean radiated acoustic intensity, and the contribution of the surface integral in equation (9) to the radiated power. A vector Poisson equation can be obtained for it, however, and it thus can be determined by using well-known magnetostatic methods, similar to the electrostatic methods used to determine the scalar potential. The radiated mean acoustic intensity thus can be determined, without having to determine H' except on the surface S_E. The source terms for these Poisson equations are not necessarily complicated. For many flows of interest, for example, the dominant term of q_J is the Coriolis acceleration one, $\overline{m_i' \; (\omega' \times \mathbf{m}')_i}$ where ω is the Beltrami vorticity (the vorticity divided by the mass density). In the developing part of the turbulent mixing region of a jet, where the vorticity direction is predominantly circumferential, this becomes simply $-\overline{m_1' \; \omega_3' m_2'}$ where 1 denotes the mean momentum direction, 3 the circumferential direction normal to it, and 2 accordingly the direction outwards towards the jet boundary, normal to both the others. By experimental and theoretical means, it should not be too difficult to quantify such an expression. The use of H' as a dependent variable in aeroacoustics problems thus could bring both theoretical and practical benefits, even though it is not the sound, which is what one hears, the fluctuating pressure p'. $\bar{\rho} H'$ can be p' outside a disturbed flow region, but inside, such a component could be augmented by the $(\frac{1}{2} \; \bar{\rho} v^2_i)'$ component, or a diffusive temperature component, or both.

The correspondence noted in the third from last paragraph of section 2.2 between the first four terms of the radiated power source density q_J (the right side of equation (8)) and the time average of $-m'_i \; [(\Omega \times \mathbf{v})'_i - V'_i]$ is evidence of a good consistency between the mean intensity divergence equation (8) and the inhomogeneous wave equation (10), particularly in respect to the interpretation of the inhomogeneous terms of these equations as physical volume source distributions of radiated mean power and total enthalpy fluctuations, respectively. The lack of correspondence between some of the terms in the respective source distributions indicates the possible existence of a non-radiating component of the H' field ("near field", or "pseudo-sound"). Also as previously noted, the quantity $[(\Omega \times \mathbf{v})'_i - V'_i]$ can be expected to have a predominant role in the total source strength density of H' (see again equation (10)) for many disturbed flows of interest. It therefore merits some examination. From equation (11) for V'_i and equation (A7) of the Appendix it is evident that one has, exactly,

$$(\Omega \times \mathbf{v})'_i - V'_i = -(\partial H'/\partial x_i + \partial v'_i/\partial t). \qquad (12)$$

If this substitution were to be made in the right side of equation (10), its contrived form as a convected wave equation for H' would be demolished! The fact that this could be done, however, does not demolish the formal mathematical validity of equation (10), as it stands, as a linear inhomogeneous convected wave equation for H', and the consequent physical interpretation of its right side as a unique source distribution for the H' waves as described by this equation.

The identity (12) can be used to provide some additional physical insight about $(\Omega \times \mathbf{v})'_i - V'_i$ and its role in the source distribution. First, if the fluctuating velocity were irrotational, one would have $(\Omega \times \mathbf{v})'_i - V'_i = -\partial(H' - \partial\phi'/\partial t)/\partial x_i$, where ϕ is the scalar potential of the velocity v_i. Second, in general the first source term of equation (10) is equal to $(\partial^2/\partial x_i^2)\,(H' - \partial\phi'/\partial t)$, independent of the solenoidal component of the velocity, but the last source term, $2[c^{-2}\{\dots\}\{(\Omega\times v)'_i - V'_i\}]'$, depends on both the solenoidal (vortical) and irrotational components of $\partial v'_i/\partial t$. These observations suggest formally regarding the fluctuating velocity scalar potential ϕ' as consisting of two parts: i.e.,

$$\phi' = \phi'_H + \phi'_\Omega \quad , \quad \partial\phi'_H/\partial t \equiv H'. \tag{13}$$

Then, from equations (A7) and (8), and with U'_i denoting the solenoidal component of v'_i (derivable from a vector potential $\mathbf{A}'$, say, as $U'_i = (\nabla\times \mathbf{A})_i$, with $\Omega'_i = (\nabla\times\mathbf{U})'_i$), one has

$$\partial U'_i/\partial t + (\Omega\times v)'_i - V'_i - \partial^2\phi'_\Omega/\partial x_i\,\partial t = 0. \tag{14}$$

Accordingly $\partial\phi'_\Omega/\partial t$ satisfies the Poisson equation

$$\partial^2(\partial\phi'_\Omega/\partial t)/\partial x_i^2 = \partial[(\Omega\times v)'_i - V'_i]/\partial x_i, \tag{15}$$

and the first and last source terms of equation (10) are expressible, respectively, as

$$-\partial^2(\partial\phi'_\Omega/\partial t)/\partial x_i^2 \quad , \quad -2[c^{-2}\{\dots\}\{\partial V'_i/\partial t - \partial^2\phi'_\Omega/\partial x_i\,\partial t\}]'. \tag{16}$$

Lighthill acoustic analogy type scaling of these expressions then predicts a (representative velocity)^4 dependence of the radiated H' if the source distribution is compact (its representative length scale is small compared to the acoustic wavelength) and the representative Mach number is not too large. This separation of ϕ' into ϕ'_H and ϕ'_Ω is not as artificial as it might seem. ϕ'_H is the velocity potential that would exist if the fluctuating flow were irrotational, and the flow in general were homentropic and under no external forcing; ϕ'_Ω would then satisfy the homogeneous Laplace equation and causality considerations would require it to be zero (for a problem such as that of Figure 1). Inspection of equation (10) for such a flow reveals that the only non-zero source term is then the third, $[(\partial c^{-2}/\partial t)\,Dh/Dt]'$. Note that in this flow the mean velocity need not be irrotational, and thus one could have mean vorticity, which itself nevertheless would not give rise to any H', radiated or "near field"! Thus it appears, that *in general*, the essential, *sine qua non*, quantities in the $(\Omega\times v)'_i - V'_i$ terms of the H' source distribution could be regarded as $\partial U'_i/\partial t$ and $\partial^2\phi'_\Omega/\partial x_i\,\partial t$. These quantities, however, have computational disadvantages; it can easily be seen that ϕ'_H, ϕ'_Ω and U'_i cannot be independently determined from equations (13)-(15); also, experimental identification of the U'_i, $-\partial\phi'_H/\partial x_i$ and $-\partial\phi'_\Omega/\partial x_i$ components of the fluctuating velocity v'_i would appear to be a prohibitively difficult task (certainly by presently available methods). Nonetheless, it is clear physically from equations (13) that ϕ'_H is the velocity scalar potential associated directly with H', both inside the disturbed flow region V_s and in the radiated field outside V_S, and that ϕ'_Ω is effectively confined to V_S, and associated only indirectly with H' via its dependence on $-\partial\phi'_H/\partial x_i$, evident implicitly in equation (15). Also it is clear, particularly from equation (8), that unless there is unusually strong irreversible and/or external forcing the radiated acoustic power must be determined by the relatively simple (mathematically!) quantity $\overline{m_i\;(\omega'\times m')_i}$, in which the fluctuating Beltrami vorticity ω' is a *sine qua non*.

With respect to the Coriolis acceleration source terms, $-\overline{m'_i(\omega\times m)_i} = \overline{m_i}\,(\overline{\omega'\times m'})_i$ in equation (8) and $-\partial(\omega\times m)'_i/\partial x_i$ and $2[c^{-2}\{\dots\}\{(\omega\times m)'_i\}]'$ in equation (10), one can observe that direct Lighthill acoustic analogy type scaling predicts $\rho_s\,\Omega_s\,v_s^2$ dependence of the acoustic power output (per unit far field surface area, as it were), where ρ_s, Ω_s and v_s are the representative source region mass density, vorticity and velocity. This scaling also predicts $(v_s/c_\infty)\,\Omega_s\,v_s$ dependence of the radiated H' from the first H' source term (assumed compact, and where c_∞ is the far field sound speed), and (schematically) $c_s^{-2}\{v_s^2 + \Omega_s v_s + |V_i|_s + |\partial h/\partial x_i|_s\}\,\Omega_s v_s$ from the second source term, where c_s is the representative source region sound speed. With the dissipative and external forcing and the source region temperature gradients ignored and Ω_s taken as $O(v_s)$, for simplicity, these predicted dependences are $\rho_s v_s^3$ for the power, and $(v_s/c_\infty)\, v_s^2$ for the first H' source term and $(v_s/c_s)^2\, v_s^2$ for the second. But, for jet turbulence noise, experiment has shown that such scalings are at best rather "broad brush", because the source region is largely only the turbulent mixing region, which has an axial extent of some six (or a few more) diameters from the jet exit (see Figure 1), and in this region the source strength density's representative frequency decreases with distance from the jet exit plane. The relevant scalings from an appropriate "broad brush" interpretation of experimental results are $\rho_s(v_s/c_\infty)^5\, v_s^3$ for the radiated power dependence and $(v_s/c_\infty)^2\, v_s^2$ for the H' dependence. These experimental dependences correspond to the prediction here for the second H' source (if $c_s \sim c_\infty$) but not to those of either the first H' source or the power source. If the first of expressions (16) is used, however, for the first H' source one obtains the predicted dependence $(v_s/c_\infty)^2\, v_s^2$, which agrees with the experimental results for both the far field H' and its corresponding power output. It is therefore clear that for the turbulent mixing region of a subsonic jet one must somehow have the power source term $\overline{m_i\ (\omega' \times \mathbf{m}')_i}$ scaling as $\rho_s(v_s/c_\infty)^5\, v_s^3$, not $\rho_s v_s^3$, if the source terms in equation (10) not involving $(\Omega \times \mathbf{v})'_i - V'_i$ are not important, as the success of the Lighthill acoustic analogy approach and other theoretical and experimental evidence suggests. Some further insight on these scaling questions and the nature of the fluctuating Coriolis acceleration's divergence as a source term can be obtained by writing, exactly, $\partial(\Omega \times \mathbf{v})'_i/\partial x_i = [v_i(\partial^2 v_j/\partial x_i\, \partial x_j - \partial^2 v_i/\partial x_j^2) - \Omega_i^2]'$, or, after some manipulation, as (again exactly)

$$\frac{\partial}{\partial x_i}(\Omega \times \mathbf{v})'_i = -\frac{\partial^2}{\partial x_i^2}\left(\tfrac{1}{2}v_j^2\right)' + \frac{\partial}{\partial x_i}\left(v_i\frac{\partial v_j}{\partial x_j}\right)' + \left\{\left[\tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right]^2 + \left[\tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)\right]^2 - \left(\frac{\partial v_j}{\partial x_j}\right)^2\right\}'. \quad (17)$$

Note that the sum of the squares of the rate of strain tensor and the angular velocity tensor in this expression is equal to the notationally simpler $(\partial v_i/\partial x_j)^2$. The acoustic analogy type scaling of the right side of equation (17), under the compact source assumption, gives (schematically)

$$(v_s/c_\infty)^2\, v_s^2 + (v_s/c_\infty)\,(v_s/c_s)^2\, v_s^2 + (v_s/L_s)^2 - (v_s/c_s)^4\, v_s^2. \quad (18)$$

In obtaining expression (18) it has been assumed that $\partial v_j/\partial x_j\ (= -(1/\rho)\, D\rho/Dt)$ is equal to $-(1/\rho c^2)\, Dp/Dt$ (i.e., isentropy) and $p \sim O(\frac{1}{2}\rho v^2)$; L_s is a velocity gradient length scale and the minus sign appears with the last term as a reminder that the last two terms scale the fluctuations in the difference $[(\partial v_i/\partial x_j)^2 - (\partial v_j/\partial x_j)^2]$ between two positive definite quantities. Only the first term of expression (18), the scaling of $-\partial^2(\frac{1}{2}v_j^2)'/\partial x_i^2$, appears to have the experimentally correct form. The second and fourth terms are of v_s^5 and v_s^6 orders, respectively. The third term as it stands is very much "out of order", but would be of roughly correct form if L_s were taken to be proportional to the acoustic wavelength, as it would then become proportional to $(v_s/c_s)^2\, v_s^2$. It is known from experiment, however, that the mean flow and moving eddy length scales in the respective frequency regions of the turbulent mixing layer of a subsonic jet are significantly smaller than the wavelengths. On these grounds one might think that $[(\partial v_i/\partial x_j)^2]'$ is more important than $\partial^2(\frac{1}{2}v_j^2)'/\partial x_i^2$. But in the mixing layer the eddies are not "frozen", but moving towards the quite random state they acquire in the diffusively decaying region (see Figure 1), so their transition to such a state might require a number of eddy diameters! In any case more accurate modelling is needed to answer this kind of question for specific flows.

There remains the question of why this acoustic analogy type scaling of $\overline{m_i\ (\omega' \times \mathbf{m}')_i}$ produces $\rho_s v_s^3$ instead of the correct $\rho_s(v_s/c_\infty)^5\, v_s^3$. A first answer to this question is that $\rho_s v_s^3$ can be correct for some flows, even if not for the compact turbulent mixing layers of subsonic jets; the assured exact correctness of $\overline{m_i}\ (\overline{\omega' \times \mathbf{m}'})_i$ itself does not depend on crude scaling. A second answer, of much more interest in connection with the physical nature of turbulent mixing layer flows and their acoustic radiations, is one which could be obtained by investigating appropriately detailed local models of such flows. This kind of investigation of the nature of $\overline{m_i}\ (\omega' \times \mathbf{m}')_i$, and the source terms for H' presented here, is a task for the future.

4. REFERENCES

[1] P.E. DOAK 1993 *Proceedings of Noise '93, The International Noise and Vibration Control Conference, St Petersburg, Russia, 31 May – 3 June* 1993. The physics of aeroacoustics.

[2] P.E. DOAK 1989 *Journal of Sound and Vibration* 131(1), 67-90. Momentum potential theory of energy flux carried by momentum fluctuations.

[3] M.S. HOWE 1975 *Journal of Fluid Mechanics* 71, 625-673. Contributions to the theory of aerodynamic noise, with applications to excess jet engine noise and the theory of the flute.

[4] LORD RAYLEIGH 1877 *Theory of sound* (two volumes). New York: Dover Publications, second edition, 1945 re-issue. See volume II, chapter XIX.

APPENDIX

Equation (8) is derived as follows. From the time averages of equations (3) and (1)

$$\partial(\overline{H'm'_i})/\partial x_i = -\bar{m}_i\,\partial\bar{H}/\partial x_i + \partial\,(\overline{S_{ij}v_j + \lambda\partial T/\partial x_i})/\partial x_i + \overline{\rho Q} + \overline{m_i f_i}\ . \qquad \text{(A1)}$$

From the time average of equation (2)

$$-\bar{m}_i\,\partial\bar{H}/\partial x_i = \bar{m}_i\ (\overline{\Omega \times \mathbf{v}})_i - \bar{m}_i\ (\overline{S_{ij}/\rho}) - \bar{m}_i\ (\overline{T\partial S/\partial x_i}) - \bar{m}_i\ \bar{f}_i\ , \qquad \text{(A2)}$$

so, from equations (A1) and (A2),

$$\partial(\overline{H'm'_i})/\partial x_i = \bar{m}_i\ (\overline{\Omega \times \mathbf{v}})_i + \partial(\overline{S_{ij}v_j + \lambda\partial T/\partial x_i})/\partial x_i + \overline{\rho Q}$$

$$- \bar{m}_i\ (\overline{T\partial S/\partial x_i}) - \bar{m}_i\ \overline{[(1/\rho)\,\partial S_{ij}/\partial x_j]} + \overline{m_i f_i}\ . \qquad \text{(A3)}$$

An equation for the entropy, derivable from equations (2), (3) and (5), is

$$\rho T\partial S/\partial t + \rho\,T\,v_i\,\partial S/\partial x_i - \partial(\lambda\partial T/\partial x_i)/\partial x_i = S_{ij}\,\partial v_i/\partial x_j + \rho Q. \qquad \text{(A4)}$$

This can be rearranged as

$$\rho T\partial S/\partial t + m_i\,T\partial S/\partial x_i + m_i\,(1/\rho)\,\partial S_{ij}/\partial x_j = \partial\,(S_{ij}v_j + \lambda\partial T/\partial x_i)/\partial x_i + \rho Q. \qquad \text{(A5)}$$

Hence equation (A3) can be rewritten, with $\omega = \Omega/\rho$, as

$$\partial(\overline{H'm'_i})/\partial x_i = \bar{m}_i\ (\overline{\omega' \times \mathbf{m}'})_i + (\overline{\rho T\partial S/\partial t}) + (\overline{m_i\,T\partial S/\partial x_i}) + \{\overline{m_i(1/\rho)\,\partial S_{ij}/\partial x_j}\}$$

$$\underline{- \bar{m}_i\ (\overline{T\partial S/\partial x_i}) - \bar{m}_i\ \{\overline{(1/\rho)\,\partial S_{ij}/\partial x_j}\} + \overline{m'_i f'_i}}$$

$$= \overline{m'_i\,[f'_i - (\omega' \times \bar{\mathbf{m}})_i + (T\partial S/\partial x_i)' + [(1/\rho)\,\partial S_{ij}/\partial x_j]']} + (\overline{p'\partial S'/\partial t/R})\ ,$$

which is equation (8).

Equation (10) is derived as follows. First, by using the thermodynamic relationships (5) and definitions (6), equations (1) and (2) are rewritten as

$$c^{-2}(DH/Dt - D(\tfrac{1}{2}v_i^2)/Dt) - R^{-1}\,DS/Dt + \partial v_i/\partial x_i = 0, \qquad \text{(A6)}$$

$$\partial v_i/\partial t + (\Omega\times\mathbf{v})_i + \partial H/\partial x_i - T\partial S/\partial x_i - \rho^{-1}\,\partial S_{ij}/\partial x_j = f_i, \qquad \text{(A7)}$$

where D/Dt is the material derivative, $D/Dt = \partial/\partial t + v_j\,\partial/\partial x_j$. The fluctuating parts of these can be conveniently expressed as

$$\partial v'_i/\partial x_i = \{R^{-1}\,DS/Dt - c^{-2}\,[DH/Dt - D\,(\tfrac{1}{2}v_i^2)/Dt]\}' = \{R^{-1}\,DS/Dt - c^{-2}\,Dh/Dt\}', \qquad \text{(A6)'}$$

$$\partial H'/\partial x_i + \partial v'_i/\partial t = -(\Omega\times\mathbf{v})'_i + V'_i, \qquad \text{(A7)'}$$

where V_i is as defined in equation (11). The divergence of equation (A7)' less the time derivative of equation (A6)' is

$$\frac{\partial^2 H'}{\partial x_i^2} = -\frac{\partial}{\partial x_i}\,[(\Omega\times\mathbf{v})'_i - V'_i] - \frac{\partial}{\partial t}\left[\frac{1}{R}\frac{DS}{Dt}\right]' + \left[\frac{\partial}{\partial t}\left(\frac{1}{c^2}\right)\frac{Dh}{Dt}\right]' + \left\{\frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{DH}{Dt} - \frac{D}{Dt}(\tfrac{1}{2}v_i^2)\right]\right\}'. \qquad \text{(A8)}$$

By using equation (A7) one can write

$$\begin{aligned} DH/Dt - (D/Dt)\,(\tfrac{1}{2}v_i^2) &= \partial H/\partial t + v_i\,\partial H/\partial x_i + v_i\,\{\partial H/\partial x_i + (\Omega\times\mathbf{v})_i - V_i\} - v_i v_j\,\partial v_i/\partial x_j \\ &= \partial H/\partial t + 2\,v_i\,\partial H/\partial x_i - v_i V_i - v_i v_j\,\partial v_i/\partial x_j\,, \end{aligned} \qquad \text{(A9)}$$

since $v_i\,(\Omega\times\mathbf{v})_i = 0$. Then

$$\begin{aligned} (\partial/\partial t)\,[DH/Dt + 2\,v_i\,\partial H/\partial x_i - v_i v_j\,\partial v_i/\partial x_j - v_i V_i)]' &= [\partial^2 H'/\partial t^2 + 2\,v_i\,\partial^2 H'/\partial x_i\,\partial t - v_i v_j\,\partial^2 v_i/\partial x_j\,\partial t \\ &\quad + 2\,(\partial v_i/\partial t)\,\partial H/\partial x_i - (\partial/\partial t)\,(v_i v_j)\,\partial v_i/\partial x_j - (\partial/\partial t)\,(v_i V_i)]' \\ &= [\partial^2 H'/\partial t^2 + 2\,v_i\,\partial^2 H'/\partial x_i\partial t + v_i v_j\,\partial^2 H'/\partial x_i\,\partial x_j + v_i v_j\,(\partial/\partial x_j)\,\{(\Omega\times\mathbf{v})'_i - V'_i\} \\ &\quad + 2\,(\partial v_i/\partial t)\,\partial H/\partial x_i - (\partial v_i/\partial t)\,v_j\,\partial v_i/\partial x_j - v_i\,(\partial v_j/\partial t)\partial v_i/\partial x_j - (\partial v_i/\partial t)\,V_i - v_i\,\partial V_i/\partial t\,]'. \end{aligned} \qquad \text{(A10)}$$

Then, from equation (A7)' and carrying out further indicated differentiations, expression (A10) becomes

$$\begin{aligned} \ldots = [\partial^2 H'/\partial t^2 + 2\,v_i\,\partial^2 H'/\partial x_i\,\partial t + v_i v_j\,\partial^2 H'/\partial x_i\,\partial x_j - v_i\,\partial V_i/\partial t + v_i v_j\,(\partial/\partial x_j)\,\{(\Omega\times\mathbf{v})'_i - V'_i\} \\ + (\partial v_i/\partial t)\,(2\,\partial H/\partial x_i - V_i - v_j\,\partial v_i/\partial x_j - v_j\,\partial v_j/\partial x_i)]', \end{aligned} \qquad \text{(A11)}$$

after regrouping and interchanging i, j in $v_i\ (\partial v_j/\partial t)\,\partial v_i/\partial x_j$. Then one can write

$$(\partial v_i/\partial t)\ (2\,\partial H/\partial x_i - V_i - v_j\,\partial v_i/\partial x_j - v_j\,\partial v_j/\partial x_i) = (\partial v_i/\partial t)\ \{2\,\partial h/\partial x_i - V_i - (\Omega\times\mathbf{v})_i\} \qquad \text{(A12)}$$

by using $H = h + \tfrac{1}{2}v_i^2$ and $v_j\,\partial v_i/\partial x_j = (\Omega\times\mathbf{v})_i + \partial(\tfrac{1}{2}v_j^2)/\partial x_i$, and in turn expression (A11) can be rewritten, by using equation (A7)', as

$$\begin{aligned} \ldots &= -\ [\partial H'/\partial x_i + (\Omega\times\mathbf{v})'_i - V'_i]\,\{2\,\partial h/\partial x_i - [(\Omega\times\mathbf{v})_i + V_i]\} \\ &= (\partial H'/\partial x_i)\,\{(\Omega\times\mathbf{v})_i + V_i - 2\,\partial h/\partial x_i\} + \{(\Omega\times\mathbf{v})_i + V_i - 2\,\partial h/\partial x_i)\}\,\{(\Omega\times\mathbf{v})'_i - V'_i\}. \end{aligned} \qquad \text{(A13)}$$

Using expression (A13) in expression (A11) and the result in equation (A10) then gives, with regrouping of terms,

$$\begin{aligned} \frac{\partial}{\partial t}\left[\frac{\partial H}{\partial t} + 2\,v_i\frac{\partial H}{\partial x_i} - v_i v_j\,\frac{\partial v_i}{\partial x_j} - v_i V_i\right]' &= \left[\frac{\partial^2 H'}{\partial t^2} + 2\,\left\{v_i\frac{\partial}{\partial t} + \tfrac{1}{2}(\Omega\times\mathbf{v})_i + \tfrac{1}{2}V_i - \frac{\partial h}{\partial x_i}\right\}\frac{\partial H'}{\partial x_i} + v_i v_j\,\frac{\partial^2 H'}{\partial x_i\,\partial x_j}\right] \\ &\quad + 2\left[\left\{\tfrac{1}{2}v_i v_j\,\frac{\partial}{\partial x_j} + \tfrac{1}{2}(\Omega\times\mathbf{v})_i + \tfrac{1}{2}\,V_i - \frac{\partial h}{\partial x_i}\right\}\left\{(\Omega\times\mathbf{v})'_i - V'_i\right\}\right]' - \left[v_i\frac{\partial V_i}{\partial t}\right]', \end{aligned}$$

and inserting this expression via equation (A9) back into equation (A8) yields equation (10).