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# ARRAY DIRECTIVITY, SEA WATER INHOMOGENEITY AND THE PLANE WAVE SPECTRUM

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## INTRODUCTION

The sea is not homogeneous. Features of the underwater environment include internal waves, stratification, turbulence, fronts, and eddies, all of which contribute to a random structure which induces fluctuations in acoustic signals. In this paper, the effect of these fluctuations on the performance of an array is considered.

This is a topic which has received little attention in the open literature. Many authors, eg [1,2], have investigated the effects of entirely random phase and amplitude variations, but in the present case the fluctuations are partially correlated across the array. Statistically derived expressions for the time-averaged beampattern [3,4] of an array in the presence of correlated phase and amplitude fluctuations, as well as its variance [5], have been given previously, but these results are limited to uniformly spaced line arrays with no shading or steering. More recently, the author has described a simulation technique [6,7] which can be applied to any arbitrary array geometry with any form of shading or steering. Such simulations, however, do not give any insight into the mechanisms in operation, and it is often difficult to explain the results obtained.

The present paper presents a new method of obtaining the average directivity pattern of an array, subject to correlated phase and amplitude fluctuations, which is based on the angular plane wave spectrum of the fluctuating wave field. The method is applicable to any array geometry, shading or steering, and in the special case of a uniform line array leads to the same equations as the previous statistical methods [3,4]. Futhermore, because the angular spectrum approach gives greater understanding of the problem, it becomes possible to derive simplified expressions for the minimum beamwidth achievable in a given propagation environment.

## AN ANGULAR SPECTRUM OF PLANE WAVES

It is convenient to think of a fluctuating wavefront as made up of a series of uniform plane waves travelling in different directions, known as the plane wave spectrum. Consider a single plane wave from this spectrum propagating in the L,r plane with its wave normal making an angle $\theta$ with the L axis, as sketched in Figure 1. L is the mean direction of propagation and r is the transverse direction. Let the complex amplitude of the single plane wave be $A(\theta)$, so the field at the point (L,r) may be written

$$u(L,r) = A(\theta)\exp(ik(r\sin\theta + L\cos\theta)), \qquad (1)$$

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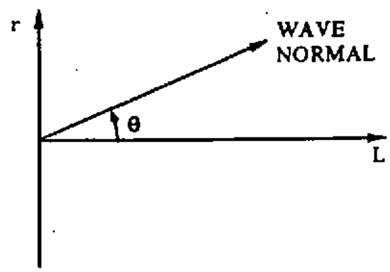


Fig. 1. Wave normal of a single component of the plane wave spectrum.

k being the wavenumber; $k = 2\pi f/c$ where f is frequency and c is sound speed. This idea can be extended to a bundle of waves at angles between θ and θ + dθ and a resultant complex amplitude A(θ)dθ. The corresponding field at (L,r) is given by

$$u(L,r) = A(\theta)d\theta\exp\{ik(r\sin\theta + L\cos\theta)\}. \quad (2)$$

The resulting component of the field in the r direction due to all the waves for θ in the forward half plane can then be written

$$u(L,r) = \int_{-\pi/2}^{\pi/2} A(\theta)\cos\theta\exp\{ik(r\sin\theta + L\cos\theta)\}d\theta. \quad (3)$$

The angle θ may be expressed in terms of $s = \sin\theta$, so that the complex amplitude is transformed to some function F(s), which is actually the plane wave spectrum, and Equation (3) becomes

$$u(L,r) = \int_{-1}^{+1} F(s)\exp\{ik(rs + L\cos\theta)\}ds. \quad (4)$$

This equation represents a wave field varying over two dimensions L and r in terms the plane wave spectrum F(s). Over the plane L = 0, the wave field is given by

$$u(r) = \int_{-\infty}^{+\infty} F(s)\exp\{ikrs\}ds, \quad (5)$$

which is immediately recognisable as a Fourier transform. It is noted that the limits of integration have been increased from ±1 to ±∞, but |s| > 1 represents imaginary values of θ which, in turn, represent evanescent waves. These are attenuated so rapidly that their contribution may be neglected. Thus the plane wave spectrum and the complex amplitude of the wavefield along a line normal to the mean propagation direction form a Fourier transform pair.

## THE AVERAGE BEAMPATTERN

The plane wave spectrum gives the distribution of the acoustic field as a function of angle, whilst a receiving array observes this field with a finite angular resolution determined by its directivity pattern. The array output is then the convolution of the plane wave spectrum and the array directivity pattern [7], given by the convolution integral:

$$Y(s) = W(s)*F(s) = \int_{-\infty}^{+\infty} W(s')F(s - s')ds', \tag{6}$$

where W(s) is the unperturbed array directivity pattern, F(s) is the plane wave spectrum of the wave field and Y(s) is the resultant array directivity. If the plane wave spectrum represents a fluctuating wavefield then Y(s) is the average beampattern.

At this point, the spatial frequency $\nu$ is introduced, where

$$\nu = k\sin\theta. \tag{7}$$

It is then pointed out that, although the field described by u(r) in Equation (5) is not known specifically, the square of the plane wave spectrum (ie the power spectrum) is obtained in terms of $\nu$ from the Fourier transform of the mutual coherence function of the wave field [8]. This is simply because the mutual coherence function is just the complex spatial correlation function of the field; correlation functions and power spectra in the relevant domains are related by the Fourier transform [9]. Furthermore, the array directivity pattern in terms of $\nu$ is just the Fourier transform of the distribution of sensitivity across the array [10].

Another well known theorem relating to Fourier transforms [9] may now be noted, that is that the convolution of two functions is the transform of the product of their transforms. Thus, the expected mean square beampattern is given by the transform of the product of the mutual coherence function and the autocorrelation of the array sensitivity distribution, written as

$$\mathbf{E}\left[\langle Y^2\rangle\right] = \int_{-\infty}^{+\infty} \Gamma(r)\left[\int_{-\infty}^{+\infty} T(\eta)T(\eta + r)d\eta\right]e^{-i\nu r}\,dr, \tag{8}$$

where $\Gamma(r)$ is the coherence function and T(r) is the array sensitivity distribution.

It should be clear that Equation (8) may be applied to any line array or continuous aperture. The element spacings in a line array need not be uniform, the elements need not be point sources, and any shading or steering function may be incorporated (T(r) may be complex). Furthermore, if a two or three dimensional Fourier transform is used, the method can be extended to planar and conformal arrays. Nevertheless, in order to provide illustrative examples, attention will now be restricted to a uniform line array of N elements and spacing d. The sensitivity distribution T(r) is then expressible as a series of unit impulses, and it may be shown that its autocorrelation function is given by [7]

$$\int_{-\infty}^{+\infty} T(\eta)T(\eta + r)d\eta = \sum_{n=-(N-1)}^{N-1} (N - n)\delta(r - nd), \tag{9}$$

where $\delta$ is the impulse function. Thus, Equation (8), may be written as

$$\mathbf{E}\left[\langle Y^2\rangle\right] = \int_{-\infty}^{+\infty} \Gamma(r)\left[\sum_{n=-(N-1)}^{N-1} (N - n)\delta(r - nd)\right]e^{-ivr}dr. \tag{10}$$

Both the coherence function and the autocorrelation function are real and even, so the Fourier transform may be replaced by the cosine transform and, using the sifting properties of the impulse function [9], Equation (10) becomes

$$\mathbf{E}\left[\langle Y^2\rangle\right] = N + 2\sum_{n=1}^{N-1} (N - n)\Gamma(nd)\cos(vnd). \tag{11}$$

Finally, some tidying up may be carried out for consistency with the earlier literature: the entire expression is normalised by dividing by $N^2$ and the variable $\gamma = kd\sin\theta$ is introduced. The end result is

$$\mathbf{E}\left[\langle Y^2\rangle\right] = \frac{2}{N^2}\left\{\frac{N}{2} + \sum_{n=1}^{N-1} (N - n)\Gamma(nd)\cos(\gamma n)\right\}. \tag{12}$$

## PHASE AND AMPLITUDE FLUCTUATIONS

Equation (12) gives the expected mean square beampattern of an array in terms of the mutual coherence function of the fluctuating wave field, a property often measured in experiments. Much of the theory of propagation in random media, however, gives results in the form of variances and spatial correlation functions for the phase and amplitude fluctuations independently. The coherence function may be obtained from these by using the relationship [8]

$$\Gamma(r) = \exp\left[\langle S^2\rangle(C_S(r) - 1) + \langle\chi^2\rangle(C_\chi(r) - 1)\right], \tag{13}$$

where $S$ is phase, $\chi$ is log-amplitude and $C_S$ and $C_\chi$ are the phase and log-amplitude correlation functions. Equation (12), may now be written as

$$\mathbf{E}\left[\langle Y^2\rangle\right] = \frac{2}{N^2}\left\{\frac{N}{2} + \sum_{n=1}^{N-1} (N - n)\cos(\gamma n) \times \exp\left[\langle S^2\rangle(C_S(r) - 1) + \langle\chi^2\rangle(C_\chi(r) - 1)\right]\right\}. \tag{14}$$

This is equivalent to the result obtained by Lord and Murphy [4] using statistical techniques. However, obtaining the result by the plane wave spectrum method, gives information about the behaviour of the degraded beampattern which is not obvious from the statistical approach.

## THE BEAMWIDTH LIMIT

Referring either to Equation (6) or (8), and from a consideration of the way that the convolution integral operates, it should be clear that when two functions of finite width are combined in this way the result approximates to the broader of the two, but is "smeared" over a width equivalent to the narrower of the two. Thus, if the unperturbed array directivity pattern is very much broader than the plane wave spectrum, the convolution of the two will not differ greatly from the array directivity pattern and there will be little apparent degradation.

If the directivity pattern is then made narrower until its width becomes comparable with that of the plane wave spectrum, the resulting average beampattern will be smeared considerably, sidelobes will run together and the main beam will broaden.

In the limit, when the unperturbed array pattern becomes narrower than the plane wave spectrum, the shape of the convolution will tend towards that of the plane wave spectrum. This is a most important point. The angular resolution of an array cannot be less than the width of the plane wave spectrum of the incident wave field. This represents the ultimate limit to the resolution. It is independent of the array and is determined entirely by the medium and its effect on acoustic propagation.

This point is demonstrated in Figures 2 and 3. In Figure 2 the dashed lines show the beamwidth of a uniform line array, with no signal fluctuations, plotted against the length of the array, from 0.1m to 100m on logarithmic scales. The frequency is 30kHz in Figure 2A and 100kHz in 2B and, because of the inverse relationship between beamwidth and array length, the plot appears as a straight line with a slope of -1.

In order to determine the beamwidth of degraded beampatterns, signal fluctuation parameters were derived from environmental measurements [7] (see also Equations

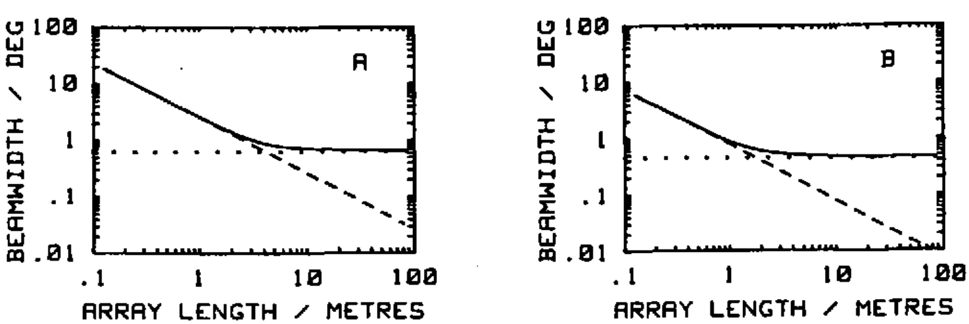


Fig. 2. Beamwidth of ideal pattern (dashed line) plotted against array length compared with average pattern for 'typical' fluctuations (solid line) and width of the plane wave spectrum (dots) for 30kHz at 1000m (A) and 100kHz at 500m (B).

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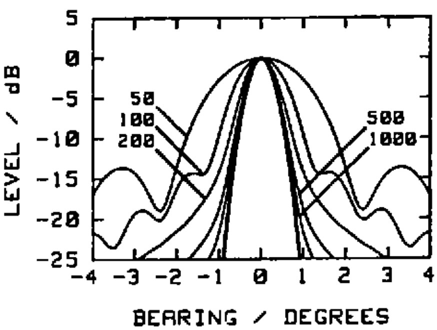


Fig. 1. Average beampatterns at 30kHz 1000m for arrays of 50, 100, 200, 500 and 1000 elements compared with the plane wave spectrum (heavy line).

(18) and (20) below) and have been used in Equation (14). The results are plotted as the solid lines and correspond to a frequency of 30kHz and propagation range of 1000m in 2A, and 100kHz at 500m in 2B. For comparison, the dotted lines show the width of the relevant plane wave spectra, derived from the Fourier transform of the coherence function as defined in Equation (13).

It can be seen that the degraded beamwidths initially follow the ideal beamwidth, but as the array length increases they settle asymptotically to the width of the plane wave spectrum. This is shown again in Figure 3 where the average patterns at 30kHz for arrays of 50, 100, 200, 500 and 1000 elements are plotted along with the plane wave spectrum, plotted as the heavy line. As the array size increases the degraded beampatterns not only tend towards the width of the plane wave spectrum, but also converge towards the same shape.

AN APPROXIMATE FORMULA FOR THE BEAMWIDTH LIMIT

Having shown that the angular width of the plane wave spectrum represents the limit to the resolution achievable by an array in an inhomogeneous medium, an approximate expression for this width will now be derived. However, it is not within the purview of this paper to delve into the complexities of random propagation theory, so a number of assumptions and approximations will be used without justification except by reference to the relevant literature.

The first of these assumptions is that phase fluctuations dominate the degradation of array beampatterns, so that amplitude fluctuations might be neglected, and Equation (13) re-written as

$$\Gamma(r) = \exp\left[\langle S^2\rangle(C_S(r) - 1)\right]. \qquad (15)$$

It is then noted that the plane wave spectrum is obtained from the Fourier transform of the coherence function, and so its angular width is inversely proportional to the width in wavelengths of the coherence function. The constant of proportionality is easily determined empirically, and if the width of the coherence function, $W_C$, is defined as the separation at which its value falls to 0.8 of maximum, then the limiting beamwidth, $\theta_{lim}$, is the half-power width of the angular spectrum, given by

$$\theta_{lim} = \lambda/4W_C, \tag{16}$$

and from Equation (5)

$$C_S(W_C) = 1 + \ln(0.8)/\langle S^2\rangle. \tag{17}$$

For a wide range of conditions to be found in the ocean, the phase correlation function $C_S$ may be approximated by [11]

$$C_S(r) = \exp(-r^2/a^2), \tag{18}$$

where a is the mean scale size of the inhomogeneities in the medium, so $W_C$ is then obtained from

$$W_C = a\left\{-\ln\left[\frac{\ln(0.8)}{\langle S^2\rangle}\right]\right\}^{1/2} \tag{19}$$

The phase variance $\langle S^2\rangle$ is given by [11]

$$\langle S^2\rangle = \frac{\sqrt{\pi}}{2}\langle\mu^2\rangle k^2 aL \tag{20}$$

where $\langle\mu^2\rangle$ is the variance of the refractive index fluctuations, generally about $1.0 \times 10^{-7}$ in the ocean, but the scale, a, may vary from less than a metre to several hundred metres, depending on the underlying mechanisms. In the computations for Figures 2 and 3, a value of 20m [7] was used for scale size a.

## DISCUSSION AND CONCLUSIONS

Equation (16) above, along with either a measurement of the mutual coherence function or knowledge of the parameters a and $\langle\mu^2\rangle$, allows a reasonable estimate of the best angular resolution achievable from a sonar array. This information obviously contributes to the selection of the most cost effective array for any particular system.

The shape of the average beampattern may then be determined from either Equation (14) or (12) for a uniform line array, or Equation (8) for the more general case.

Finally, it should be pointed out that the expected beampattern represents the angular response of a system with time averaging over a number of pings. Individual realisations of the beampattern may vary considerably, and the theory presented here says nothing about this variability. If the spatial scale of the signal fluctuation is small compared with the array, the main lobe of individual patterns will be similar to the average, and most of the variability will be in the sidelobe region. If the scale of the signal fluctuations is larger than the array, however, individual patterns will approximate to the unperturbed directivity function, but the apparent pointing direction of the main lobe will vary over roughly the width of the plane wave spectrum. This variation in apparent signal arrival direction may be significant in system design.

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