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UNDERWATER PROPAGATION MODELLING USING FINITE DIFFERENCE METHODS

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1. INTRODUCTION

Underwater acoustic wave propagation problems deal with the solution of representative partial differential equations. These equations, which govern realistic physical underwater acoustic phenomena, are all regarded as wave equations. Because of the complex nature of the sea water medium, the various wave equations can be very complicated, and permit a closed form solution only in simple cases. Thus, for simplicity, specialised approximations to these problems are considered, and a number of wave equations are in existence in different forms, each having its own advantages due to special approximations or treatments.

The approximations made generally result in a loss of accuracy or limit the range of validity of the solution. Even when these approximations permit a closed form solution it usually involves special functions, integrals, etc. which are often not convenient to evaluate. In such situations, numerical solution of the wave equation may offer significant advantages by permitting treatment of more realistic environments. The model described in this paper embodies one such numerical method, and employs an implicit finite difference (IFD) scheme for solving the parabolic approximation to the wave equation. It is based largely on work published by Lee and Botseas [1].

The particular model, however, is perhaps not relevant to the main theme of this paper, which is to give an insight into the internal workings of a propagation model, the mathematics involved, and the computer implementation.

2. UNDERWATER ACOUSTIC WAVE PROPAGATION

In this Section, the general underwater acoustic propagation problem is formulated, and the approximations that will be used to reduce the problem to a computationally tractable form are introduced. The limitations that these approximations impose on the solution and the conditions under which they are valid are discussed.

2.1 The Governing Wave Equation

Acoustic propagation in the ocean medium for a harmonic point source is governed by the reduced wave equation, a homogeneous Helmholtz equation, of the general form

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$$\nabla^2 p + k_o^2 n^2 p = 0, \tag{1}$$

where k_o is the reference wavenumber, ω/c_o,
$n = n(r,z)$ is the refractive index, $c_o/c(r,z)$,
p is the acoustic pressure,
∇^2 is the Laplacian operator,
c_o is the reference sound speed,
$c(r,z)$ is the sound speed at range r and depth z,
f is the source frequency and $\omega = 2\pi f$.

Because there is rarely sufficient oceanographic information available to warrant a full three-dimensional solution to the wave equation, the first approximation will be to consider a two-dimensional version. To include the radial spreading experienced by a point source, cylindrical coordinates must be used, and Equation (1) takes the form

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + k_o^2 n^2 p = 0. \tag{2}$$

However, firstly because of the lack of oceanographic information mentioned above and, secondly, because only the plane containing both source and receiver is generally of interest, the azimuthal variation may be neglected. The two dimensional form of Equation (1) can then be written as

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k_o^2 n^2 p = 0. \tag{3}$$

The solution of Equation (3), with its relevant boundary conditions, requires certain environmental information, particularly the sound velocity profile (SVP), and surface and bottom conditions.

When the ocean medium (possibly including the sea bed) is uniformly stratified, ie. when the density, refractive index and thickness of each layer are range invariant, the problem is defined as range-independent. If the ocean environment is not stratified uniformly, or if the physical properties (particularly the SVP or the water depth) vary with range, the problem is range-dependent. In this paper the environment is assumed range-dependent, although solutions to range-independent problems are always available as a subset.

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2.2 The Parabolic Approximation

In most ocean environments, long range, low frequency sound propagation is dominated by rays having small grazing angles since rays propagating at steep angles are attenuated by penetration and absorption in the sea bed. For efficient handling of this class of problems, Tappert [2] introduced the parabolic equation (PE) approximation method which decomposes the elliptic wave equation into two equations through the choice of an arbitrary separation constant.

The method begins by expressing the acoustic pressure, $p(r,z)$, in Equation (3) as $p(r,z) = u(r,z)v(r)$, where $v(r)$ is strongly dependent on r while $u(r,z)$ depends only weakly on r. This expression may be substituted into Equation (3) to obtain

$$u\left[v_{rr} + \frac{1}{r}v_r\right] + v\left[u_{rr} + u_{zz} + \left(\frac{1}{r} + \frac{2}{v}v_r\right)u_r + k_o^2 n^2 u\right] = 0. \qquad (4)$$

Note the use of subscripts as shorthand for partial derivatives. Now, in order to cause the left hand side of Equation (4) to vanish, if k_o is used as a separation constant, it is necessary that

$$\left[v_{rr} + \frac{1}{r}v_r\right] = -k_o^2 v \qquad (5)$$

and

$$\left[u_{rr} + u_{zz} + \left(\frac{1}{r} + \frac{2}{v}v_r\right)u_r + k_o^2 n^2 u\right] = k_o^2 u. \qquad (6)$$

Upon rearrangement, Equations (5) and (6) become

$$v_{rr} + \frac{1}{r}v_r + k_o^2 v = 0 \qquad (7)$$

and

$$u_{rr} + u_{zz} + \left(\frac{1}{r} + \frac{2}{v}v_r\right)u_r + k_o^2\left(n^2 - 1\right)u = 0. \qquad (8)$$

Equation (7) is a second order ordinary differential equation, whose solution for an outgoing wave is simply

$$v(r) = H_o^{(1)}(k_o r), \qquad (9)$$

where $H_o^{(1)}(k_o r)$ is the zero order Hankel function of the first kind.
Thus, once the field for u has been found it is only necessary to multiply by the Hankel function to reassemble the sound pressure field.

Applying the far field approximation, $k_o r \gg 1$, Equation (9) can be approximated by

$$v(r) = \left(\frac{2}{\pi k_o r} \right)^{1/2} \exp\left[i \left(k_o r - \frac{\pi}{4} \right) \right]. \qquad (10)$$

Using this expression to simplify the coefficient $(1/r + (2/v)v_r)$ in Equation (8) leads to

$$u_{rr} + u_{zz} + 2ik_o u_r + k_o^2 \left[n^2(r,z) - 1 \right] u = 0. \qquad (11)$$

Finally, if the fractional change in u_r over a wavelength is small, then $|u_{rr}| < |2ik_o u_r|$. This inequality, the paraxial approximation, allows the u_{rr} term in Equation (11) to be dropped and, with some rearrangement, gives

$$u_r = \frac{i}{2} k_o \left[n^2(r,z) - 1 \right] u + \frac{i}{2k_o} u_{zz}. \qquad (12)$$

This is the parabolic equation (PE) introduced by Tappert [2].

2.3 Validity and Limitations

Equation (12) above is the conventional PE, widely used to predict transmission loss in underwater acoustics. It should be stressed at this point that this is an approximation to the full wave equation, embodying simplifications that allow a solution to be obtained within the limits of current computing capabilities. One major limitation that is not immediately clear (although it is obvious in more formal derivations [3]) is that the back-scattered field is neglected; the approach is thus usable only for propagation loss predictions, and cannot, for example, model reverberation.

Using the parabolic approximation imposes another important limitation: the validity of the approximation is restricted to propagation at angles close to the horizontal. It is noted that the PE, Equation (12), may be expressed in the form

$$\frac{\partial u}{\partial r} = ik_o\left(\frac{A + Bq}{C + Dq} - 1\right)u, \qquad (13)$$

where q is some differential operator involving k_o and n, and the coefficients A, B, C and D, for the Tappert equation, are 1, 1/2, 1 and 0 respectively. By comparing results with those obtained from the full wave equation [4], it may be shown that the PE in the form of Equation (12) or (13), and using the Tappert coefficients, is a good approximation provided the angle of propagation is less than 15°.

Alternative formulations of the PE are possible. In one version due to Claerbout [5], the coefficients A, B, C and D are 1, 3/4, 1 and 1/4 respectively, whilst Greene [6], by minimising the approximation error, obtained values of A = 1.430463, B = 1.139144, C = 1.430648 and D = 0.430648. Comparison with the wave equation shows that the Claerbout and Greene equations are valid for propagation angles up to about 40°. Noting that the Tappert equation is the tried and tested method, most variants of the IFD program allow the user to implement any version of the PE, provided that the coefficients A, B, C and D are known.

3. THE FINITE DIFFERENCE MODEL

The finite difference method (not to be confused with finite element methods) is a general purpose numerical scheme for solving partial differential equations and whose theory and applications have been described clearly in many text books (eg. [7]). However, for those not familiar with these methods, a brief explanation of the basic principle may be given. That basic principle is that derivatives at a point may be approximated by difference quotients over a small interval, ie. $\partial\phi/\partial x$ is replaced by $\Delta\phi/\Delta x$, where Δx is small, as follows:

3.1 Finite Difference Approximations to Derivatives

Referring to Figure 1, when a function f(x) and its derivatives are single-valued, finite and continuous functions of x, then by Taylor's theorem

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2f''(x) + \frac{1}{6}h^3f'''(x) + \ldots \qquad (14)$$

and

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2f''(x) - \frac{1}{6}h^3f'''(x) + \ldots \qquad (15)$$

where, to add to the confusion, derivatives are now denoted by primes. Addition of these equations gives

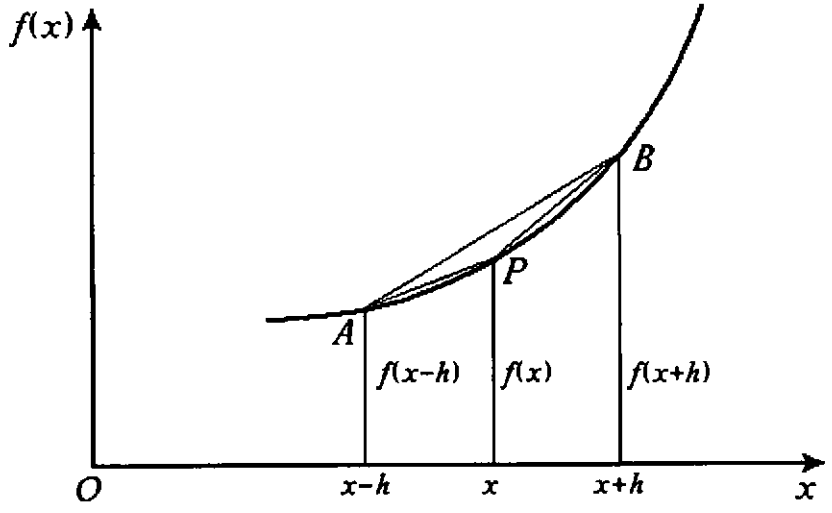

Fig.1 Sketch showing notation used in deriving finite difference approximations to derivatives.

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^4), \qquad (16)$$

where $O(h^4)$ means terms in fourth and higher powers of h. Assuming that these are negligible in comparison with lower powers of h it follows that

$$f''(x) \simeq \frac{1}{h^2}\{f(x+h) - 2f(x) + f(x-h)\}, \qquad (17)$$

with an error on the right hand side of order h^2.

Subtraction of Equation (15) from Equation (14) and neglect of terms of order h^3 leads to

$$f'(x) \simeq \frac{1}{2h}\{f(x+h) - f(x-h)\}, \qquad (18)$$

with an error of order h^2.

Equation (18) clearly approximates the slope of the tangent at P by the slope of the chord AB, and is called the central-difference approximation. It is also possible to approximate the slope of the tangent at P by either the slope of the chord PB, giving the forward-difference formula

$$f'(x) \simeq \frac{1}{h}\{f(x+h) - f(x)\}, \tag{19}$$

or the slope of the chord AP, giving the backward-difference formula

$$f'(x) \simeq \frac{1}{h}\{f(x) - f(x-h)\}. \tag{20}$$

Equations (19) and (20) can both be written down immediately from Equations (14) and (15) respectively, assuming second and higher powers of h are negligible. This shows that the errors in these forward and backward difference formulae are both O(h).

In a similar manner, difference equations may be derived for higher order derivatives, and it is important to note that the accuracy of these approximations is known and predictable. It should be obvious now that differential equations can be converted to a discrete numerical form by substituting the relevant difference formulae for the derivatives.

For example, if the curve shown in Figure 1 were governed by a first order ordinary differential equation, the backward-difference could be used and the value of f(x) calculated from the known value of f(x-h), which may be an initial condition. This would be an **explicit** relationship. If the central difference formula were to be used the calculation would include f(x+h) which may, as yet, be unknown. The problem then requires further information, such as boundary conditions, for its solution. This would be an **implicit** relationship, requiring more computational effort, but is potentially more accurate than the explicit form.

Matters become more complicated when higher order derivatives and functions of several variables are considered, as is the case with the wave equation, but the principles remain the same.

The finite difference formulation of the parabolic equation may now be considered, but before proceeding it is necessary to define some symbols and concepts.

3.2 Some Definitions

The propagation domain is considered to be rectangular, and lies in the vertical plane which includes the source and receiver. This domain is partitioned into a set of small rectangular blocks, as shown in Figure 2. The index m is used to indicate the vertical direction, and the index n to indicate the horizontal direction. The wave field, u, is a function of (r,z). At the point (n,m) this means $u(r,z) = u(n\Delta r, m\Delta z)$, normally written as $u_{n,m}$. A lower case k is used to indicate Δr and a lower case h to indicate Δz.

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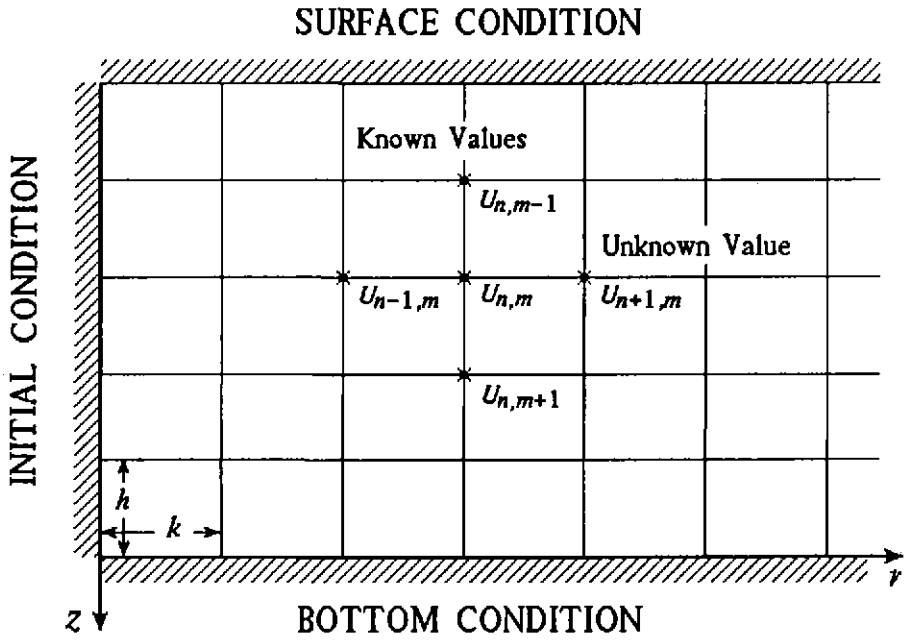

Fig.2 The rectangular propagation domain used in the IFD model.

The forward finite difference of the wave field is the difference between two neighbouring points, eg. $u_{n+1,m} - u_{n,m}$; in the reverse direction it is the backward difference, eg. $u_{n-1,m} - u_{n,m}$. By the central difference is meant $u_{n+1,m} - u_{n-1,m}$ as the forward central difference and $u_{n-1,m} - u_{n+1,m}$ as the backward central difference; both these differences regard $u_{n,m}$ as the central point. The central difference operator in the z-direction is represented by δ_z, and D is the general finite difference operator. Mathematically, δ_z operating on a function f(z) means

$$\delta_z f = f(z + h/2) - f(z - h/2). \qquad (21)$$

To establish a relationship between the two operators δ_z and D, by expanding f(z + h/2) and f(z - h/2) in powers of h, it may be shown that

$$\delta_z f = 2\sinh\left(\frac{h}{2}\frac{\partial}{\partial z}\right) f. \qquad (22)$$

The general finite difference operator D simply represents $\partial/\partial z$ in the z direction, so

$$\delta_z f = 2\sinh\left(\frac{h}{2} D\right) f, \tag{23}$$

and by using an expansion for $\sinh^{-1}$

$$D^2 = \frac{\delta_z^2}{h^2}\left(1 - \frac{1}{12}\delta_z^2 + \frac{1}{90}\delta_z^4 - \cdots\right). \tag{24}$$

In addition, the following formulae are required:

$$\alpha_m^n = 1 - (1/2)ka_m^n, \tag{25a}$$

$$\beta_m^n = b_m^n k/h^2, \tag{25b}$$

$$\gamma_m^n = 1 + (1/2)ka_m^n, \tag{25c}$$

$$X_m^{n+1} = 1 - (1/2)ka_m^{n+1} + b_m^{n+1}s, \tag{25d}$$

$$Y_m^n = \gamma_m^n - \beta_m^n = 1 + (1/2)ka_m^n - b_m^n s. \tag{25e}$$

3.3 Finite Difference Formulation of the Parabolic Equation

The standard parabolic wave equation, as derived in Section 2.2, may be written in the following general form:

$$\frac{\partial u}{\partial r} = a(k_o,r,z)u + b(k_o,r,z)u_{zz},$$

$$= Lu, \tag{26}$$

where

$$L = a(k_o,r,z) + b(k_o,r,z)\frac{\partial^2}{\partial z^2}, \tag{27}$$

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$$a(k_o,r,z) = \frac{i}{2} k_o\left(n^2(r,z) - 1\right), \qquad (28)$$

and

$$b(k_o,r,z) = \frac{i}{2k_o}. \qquad (29)$$

A Taylor expansion for $u(r + k,z)$ then leads to

$$u(r + k,z) = \left(1 + k\frac{\partial}{\partial r} + \frac{1}{2!}k^2\frac{\partial^2}{\partial r^2} + \cdots\right)u(r,z),$$

$$= \exp\left(k\frac{\partial}{\partial r}\right)u(r,z). \qquad (30)$$

By writing $z = mh$, $r = nk$ and $u(r,z) = u(nk,mh) = u_{n,m}$, it is possible to use Equation (30) to solve Equation (26) and retain only the second order difference. Thus, an explicit formula is obtained:

$$u_{n+1,m} = \left(1 + k\frac{\partial}{\partial r}\right)u_{n,m} = \left(1 + a(k_o,r,z)k + \frac{b}{h_2}k\partial_z^2\right)u_{n,m}. \qquad (31)$$

Using the second order central difference for ∂_z^2 in Equation (31) gives

$$u_{n+1,m} = (1 + a_{n,m}k)u_{n,m} + \frac{b_{n,m}}{h^2}k(u_{n,m+1} - 2u_{n,m} + u_{n,m-1}). \qquad (32)$$

This is an explicit finite difference equation by which the field values at a range $r = (n + 1)k$ can be computed entirely from the known values at the previous range step, $r = nk$.

Although the explicit method is computationally simple it has one serious drawback. The range step $\Delta r = k$ is necessarily very small because the process is valid only for $0 < k/h^2 \leq 1/2$, ie. $k \leq h^2/2$, and $h = \Delta z$ must be kept small in order to attain reasonable accuracy. Crank and Nicolson [8] proposed a

method that reduces the total volume of calculation and is valid (ie. convergent and stable) for all finite values of k/h^2. The method starts by splitting Equation (30) in such a way that

$$\exp\left(-\frac{1}{2}k\frac{\partial}{\partial r}\right)u_{n+1,m} = \exp\left(\frac{1}{2}k\frac{\partial}{\partial r}\right)u_{n,m} \qquad (33)$$

Expanding the exponential series on both sides of expression (33) and retaining only the linear terms leads to

$$\left(1 - \frac{1}{2}k\frac{\partial}{\partial r}\right)u_{n+1,m} = \left(1 + \frac{1}{2}k\frac{\partial}{\partial r}\right)u_{n,m}. \qquad (34)$$

Since $\partial/\partial r = a(k_o,r,z) + b(k_o,r,z)\partial^2/\partial z^2$, Equation (34) becomes

$$\left[1 - \frac{1}{2}k\left(a(k_o,r,z) + b(k_o,r,z)\frac{\partial^2}{\partial z^2}\right)\right]u_{n+1,m}$$

$$= \left[1 + \frac{1}{2}k\left(a(k_o,r,z) + b(k_o,r,z)\frac{\partial_2}{\partial z^2}\right)\right]u_{n,m}. \qquad (35)$$

Using the first term of Equation (24) for D^2, and substituting into Equation (35) gives

$$\left[1 - \frac{1}{2}k\left(a(k_o,r,z) + b(k_o,r,z)\frac{1}{h^2}\delta_z^2\right)\right]u_{n+1,m}$$

$$= \left[1 + \frac{1}{2}k\left(a(k_o,r,z) + b(k_o,r,z)\frac{1}{h^2}\delta_z^2\right)\right]u_{n,m} \qquad (36)$$

Substituting for δ_z^2 in Equation (36) and writing $s = k/h_2$ produces

$$\left[1 - (1/2)k\left(a(k_o,r,z) + b(k_o,r,z)s\right)\right]u_{n+1,m} - (1/2)b(k_o,r,z)s\left(u_{n+1,m+1} + u_{n+1,m-1}\right)$$

$$= \left[1 + (1/2)k\left(a(k_o,r,z) + b(k_o,r,z)s\right)\right]u_{n,m} + (1/2)b(k_o,r,z)s\left(u_{n,m+1} + u_{n,m-1}\right). \qquad (37)$$

Equation (37) is an implicit finite difference equation known usually as the Crank-Nicolson scheme.

The parabolic equation has now been formulated as a finite difference equation in two ways: an explicit scheme (Equation (32)), and an implicit scheme (Equation (37)). The explicit scheme does not need information on the next (n+1) range level but generally requires a small step size for stability. The implicit scheme, on the other hand, is unconditionally stable. Despite the dependence of Equation (37) on the field at the next (n+1) range level, this equation may be solved given the field at the previous range only and the necessary boundary conditions at the surface and bottom. From here on attention will be confined to the implicit formulation; explicit schemes are not viable for this type of application because of the small step size required (but see [4] for some conditionally stable explicit schemes).

Equation (37) is particularly amenable to numerical computation, using matrix methods. It is clear that to solve the parabolic equation the field must be known at some initial range, and that the appropriate boundary conditions must be imposed at the surface and bottom of the rectangular propagation domain. Boundary information is needed at both the present (n) and next (n+1) range levels, described by $u(r_0,z_0)$, $u(r_1,z_0)$, $u(r_0,z_B)$ and $u(r_1,z_B)$. The first two points are the surface boundary points, and the last two are the bottom boundary points.

Now, the relationship between D^2 and δ_z^2 has already been given (Equation (24)) so, using the definitions given in Equations (25a)-(25e), Equation (37) can be expressed in matrix form:

$$\begin{bmatrix} X_1 & -\frac{1}{2}\beta_1^{n+1} & 0 & \cdots & 0 & 0 \\ -\frac{1}{2}\beta_2^{n+1} & X_2 & -\frac{1}{2}\beta_2^{n+1} & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & X_{m-1} & -\frac{1}{2}\beta_{m-1}^{n+1} \\ 0 & 0 & 0 & \cdots & -\frac{1}{2}\beta_m^{n+1} & X_m \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{m-1}^{n+1} \\ u_m^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\beta_1^{n+1}u_o^{n+1} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}\beta_m^{n+1}u_{m+1}^{n+1} \end{bmatrix}$$

$$+ \begin{bmatrix} Y_1 & \frac{1}{2}\beta_1^{n} & 0 & \cdots & 0 & 0 \\ \frac{1}{2}\beta_2^{n} & Y_2 & \frac{1}{2}\beta_2^{n} & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & Y_{m-1} & \frac{1}{2}\beta_{m-1}^{n} \\ 0 & 0 & 0 & \cdots & \frac{1}{2}\beta_m^{n} & Y_m \end{bmatrix} \begin{bmatrix} u_1^{n} \\ u_2^{n} \\ \vdots \\ u_{m-1}^{n} \\ u_m^{n} \end{bmatrix} - \begin{bmatrix} \frac{1}{2}\beta_1^{n}u_o^{n} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}\beta_m^{n}u_{m+1}^{n} \end{bmatrix} \quad (38)$$

Equation (38) forms the basis of the IFD model. If the appropriate surface and bottom boundary values are known, along with the field at all depth points at the present range (n) - which may be the initial field - then the field at all depth points at the next range level (n+1) is found by inverting the matrix operator on the left hand side using standard techniques. The process may then be repeated to advance the computed field in range throughout the propagation domain in steps of h. The specification of the initial and boundary conditions will be considered in the next section.

3.4 Initial and Boundary Conditions, and other Complicating Factors

Because an initial boundary value problem is being solved, it is necessary to specify the field over a depth grid at some initial range to start the computational procedure. Surface and bottom boundary conditions need to be

specified, and the treatment of interfaces between layers must be considered. In addition, the effect of attenuation must be included. These matters will not be dealt with in detail, but an indication will be given of how they are incorporated in the model.

3.4.1 The Initial Field

Potentially, a number of propagation models such as ray tracing, normal mode theory or the Fast Field Program can be used to generate the starting field. For many applications, however, it is sufficient to approximate the initial field by a Gaussian function [2]. This procedure is adopted in most implementations of the IFD model, with an option for the user to insert his own starting field if he wishes. A Gaussian beam is considered, where

G_w = the 3dB beamwidth of the main lobe,

G_a = the amplitude of the beam at its peak,

z_s = the depth of the source.

The real part (the imaginary part is set to zero) of the starting field at mth mesh point in the depth direction is then given by

$$u_{0,m} = G_a\left(\exp\left[-\left(\frac{mh-z_s}{G_w}\right)^2\right] - \exp\left[-\left(\frac{-mh-z_s}{G_w}\right)\right]^2\right) \qquad (39)$$

3.4.2 Boundary Conditions

In most low frequency sound propagation applications the surface may be treated as a pressure release boundary, and the field is assumed to vanish at the surface. Specifying the bottom boundary condition is considerably more difficult, especially when limited environmental data are available. The problem may be overcome, however, by assuming that the lowest layer in the bottom extends to such a depth that the field has been reduced to a negligible level by absorption.

Absorption is included, both in the water column and in the sea bed, by making the refractive index complex and inserting the attenuation in the imaginary part, using the formula

$$\mu_m^2 = \left(\frac{c_0}{c_m}\right) + i\left(\frac{c_0}{c_m}\right)\frac{\alpha}{27.287527} \qquad (40)$$

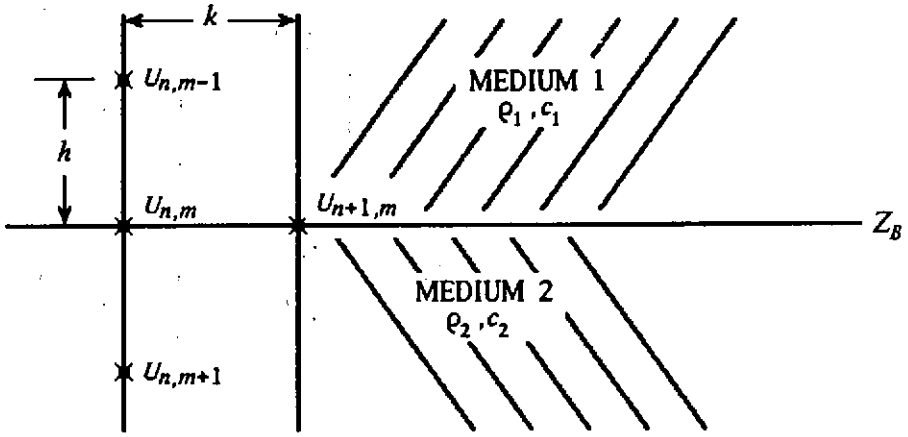

Fig.3 Treatment of a Horizontal Interface.

where α is the attenuation in dB/wavelength,

c_0 is the reference sound speed,

c_m is the sound speed at the mth depth point,

and μ_m is the refractive index at the mth depth point.

3.4.3 Interface Treatment

Another problem that arises in the practical implementation of the finite difference scheme is the treatment of the field at the interfaces between different media. The situation is shown schematically in Figure 3. The subscripts 1 and 2 denote the medium above and below the interface respectively, ρ is density and c is sound speed. Interfaces are constrained to coincide with depth grid points.

The interface requires a modification to the parabolic equation, which gives rise in turn to a change in the implicit finite difference formulation and its matrix representation (Equation (38)). The new matrix equation is

FINITE DIFFERENCE METHODS

$$\begin{bmatrix} X_1 & -\frac{\rho_1}{\rho_2} & 0 & \cdots & 0 & 0 \\ -1 & X_2 & -\frac{\rho_2}{\rho_3} & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & X_{m-1} & -\frac{\rho_{m-1}}{\rho_m} \\ 0 & 0 & 0 & \cdots & -1 & X_m \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{m-1}^{n+1} \\ u_m^{n+1} \end{bmatrix} = \begin{bmatrix} u_o^{n+1} \\ 0 \\ \vdots \\ 0 \\ u_{m+1}^{n+1} \end{bmatrix}$$

$$+ \begin{bmatrix} Y_1 & \frac{\rho_1}{\rho_2} & 0 & \cdots & 0 & 0 \\ 1 & Y_2 & \frac{\rho_2}{\rho_3} & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & Y_{m-1} & \frac{\rho_{m-1}}{\rho_m} \\ 0 & 0 & 0 & \cdots & 1 & Y_m \end{bmatrix} \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{m-1}^n \\ u_m^n \end{bmatrix} - \begin{bmatrix} u_o^n \\ 0 \\ \vdots \\ 0 \\ u_{m+1}^n \end{bmatrix} \qquad (41)$$

Here, ρ_m means the density at the mth grid point. Also, it should be noted that the definitions of the X and Y are now slightly different from those given in Equation (25); for a more complete treatment, the reader is referred to [1] or [4].

3.4.4 Range Dependent Depth Variation

A range dependent model, of course, must have the ability to handle a varying water depth. This requires that the depth grid, and hence the matrix equation, (38) or (41), changes in size as the range is advanced. This turns out to be simply a matter of housekeeping within the computer program, points being added or subtracted as required, and will be discussed further in the following section.

4. COMPUTER IMPLEMENTATION

The IFD model exists as a number of variants, and the version presented in [1] and [4] will be taken as typical and described here. Listings of the program in FORTRAN will be found in [1] and [4], so details of the coding are not given in this paper.

The model consists of a main program and 10 subroutines as shown schematically in Figure 4. Those subroutines marked with an asterisk are prepared by the user if required; the subroutine marked with a dagger may be modified by the user. Input parameters that result in control being transferred to user subroutines are as shown in the figure. For example, if input parameter ISF = 1, then control is transferred to UFIELD, the user generated initial field, rather than SFIELD, the Gaussian starting field.

A brief description of each subroutine follows.

4.1 Main Program IFD

IFD is the main program and controls execution of the various subroutines which make up the model. Initially, IFD reads selected input parameters and performs initialisation of certain variables. IFD then calls on either subroutine SFIELD or subroutine UFIELD to generate the starting field that is to be marched out in range. IFD then calls on subroutine DIAG to compute the main diagonals of the matrices that represent the system of equations at the present and advanced ranges.

After these preliminary procedures have been accomplished, IFD enters a main loop and continues to cycle in the loop until the solution has been marched out to the maximum range requested.

At each new range step, IFD determines whether or not to update the sound speed profile and/or bottom depths. If an update is performed, IFD calls on subroutine DIAG to recompute the main diagonals in the matrices. Whether or not the diagonals have been updated, IFD calls subroutine CRNK to advance the solution one range step. The solution returned by CRNK is then written in an output file. If the solution has reached the maximum range the program is terminated. If the solution has not reached the maximum range, IFD returns to the top of the main loop and repeats the above procedures.

4.2 Subroutine SFIELD

If input parameter ISF = 0, main program IFD calls on subroutine SFIELD to generate a Gaussian starting field at zero range.

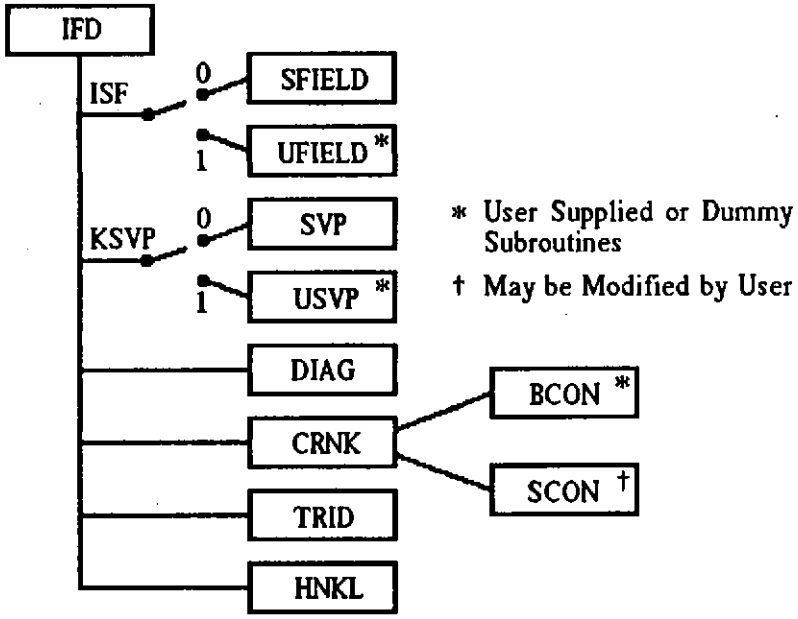

Fig.4 Structure of IFD Computer Program.

4.3 Subroutine UFIELD

If input parameter ISF = 1, main program IFD calls on the user-written subroutine UFIELD to generate the starting field.

4.4 Subroutine SVP

When the range of the solution is equal to the range of the next sound velocity (speed) profile, a new sound speed profile is read in. If input parameter KSVP = 0, subroutine SVP is called upon to read the next sound speed profile from the input data file.

At the present stage of development, linear interpolation of sound speed values is performed in depth only. Changes in the profiles with range are abrupt, with no interpolation being performed (but see under Subroutine USVP). Vertical interpolation of sound speed values is performed in subroutine DIAG.

4.5 Subroutine USVP

If input parameter KSVP ≠ 0, subroutine USVP is called to supply an updated sound speed profile at each step in range. Subroutine USVP must be prepared by the user.

Variable KSVP may be used in a computed GOTO statement to transfer control within user subroutine USVP. When the user no longer needs USVP, KSVP must be set to zero within USVP. The last profile entered will then be used until the solution range is equal to the next RSVP. If KSVP is not set to zero, then USVP will be called at each range step until the range is equal to RSVP, the range of the next profile. With this option, the user can generate a new profile at each range step, and sound speed profiles interpolated in range may be entered.

4.6 Subroutine DIAG

Subroutine DIAG computes the range-dependent and depth-dependent main diagonals of the matrices that represent the system of equations at the present and advanced solution ranges.

Prior to computing the diagonals, DIAG determines the values of sound speed, density and attenuation to be used at each depth represented by the corresponding row of each matrix. Linear interpolation in depth is performed as required.

4.7 Subroutine CRNK

Subroutine CRNK computes the right hand side of the system of equations, determines bottom type, sets up bottom conditions at the present and advanced ranges and then calls on subroutine TRID to solve the tridiagonal system of equations. If the user is supplying surface conditions, CRNK calls on SCON to provide them. If the user is supplying bottom conditions, CRNK calls on user-written subroutine BCON for these conditions.

The treatment of the bottom is complex, depending on the slope of the bottom and whether the bottom is rigid, an artificial absorbing layer or some other condition supplied by the user. A full description of the bottom treatment is given in [1] and [4].

4.8 Subroutine BCON

This is an optional user-written subroutine BCON which supplies values of the field at the bottom at the present and next advanced ranges.

4.9 Subroutine SCON

If the user wishes to supply values of the field at the surface at the present and advanced ranges, he may do so by rewriting subroutine SCON. At present, subroutine SCON sets all surface values to zero (pressure release surface).

4.10 Subroutine TRID

Subroutine TRID solves a system of N linear simultaneous equations having a tridiagonal coefficient matrix, as in Equations (38) and (41), where all entries are zero except for the main diagonal and (perhaps) the diagonals on either side of it.

4.11 Complex Function HNKL

HNKL computes the Hankel function (see Equation (9)).

5. CONCLUDING REMARKS

The IFD propagation model has been described in some detail, and the primary objective of this paper has been to give an insight into the internal workings of a propagation model, the mathematics involved, and the computer implementation. Because of this, the reasons for using such a model, the relative merits of complex and simple models, the advantages and disadvantages of this particular model, and many other relevant points have not been addressed. Such topics, however, are covered elsewhere in these proceedings.

Nevertheless, a few words about why a potential user should be interested in the mathematical details of such a model are in order. The first point is that an estimate of the model's accuracy in a specific application can only be obtained by understanding the approximations involved in going from exact solutions to the equations used in the computer code.

Secondly, models such as this are usually implemented as 'research' programs. This means that although they can be extremely flexible in their application, the user is required to provide a multitude of input parameters (setting up the input file can take longer than running the program). Some of these parameters may require subjective choices, which must be based on an understanding of the models operation. Often it may be possible to decrease the execution time or the memory requirement by a suitable parameter selection, without seriously affecting the accuracy. Unfortunately, the reverse is also true. The wrong decision may considerably reduce the accuracy with no gain in running time or memory usage.

Finally, those readers wishing to know more about the IFD model should consult [1] or [4]. These publications include FORTRAN program listings which actually seem to work! Further details of the parabolic approximation can be found in [2] and [3], and the topic has been well represented in the literature (eg JASA) in recent years. The best source of information about finite difference methods in general is a suitable text book, such as [7].

6. REFERENCES

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