A NUMERICAL METHOD FOR DETERMINING THE RESPONSE OF A SONAR TRANSDUCER

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1 INTRODUCTION

In this paper we present a numerical method for predicting the acoustic field around a finite elastic structure immersed in an infinite acoustic medium. This has a number of applications in underwater acoustics and aeronautics. In particular we are interested in determining the radiated sound pattern from a simple sonar transducer.

In Section 2 we show how this can be achieved by coupling a boundary element analysis of the exterior acoustic field to a finite element analysis of the structural displacements by matching the appropriate boundary conditions. However, it is well known that the classical integral equation formulation commonly used does not have a unique solution for certain discrete frequencies [3,4,5,7]. Consequently, it is necessary to employ a modified formulation which ensures that the solution to the integral equation is unique for all frequencies. Unfortunately, the modified formulation introduces certain computational difficulties in the form of a hyper-singular operator, and we discuss a suitable method for overcoming these computational problems.

Section 3 gives a brief description of the experimental procedure used to determine the sound pressures around the transducer. In Section 4 we demonstrate that our numerical method gives a reasonably good estimate of the observed acoustic field.

2 NUMERICAL METHODS

Let S denote the closed surface of a structure immersed in an infinite acoustic medium. We denote the interior and the exterior of S by D_- and D_+ respectively . Further, let $\underline{n}$ denote the unit normal to S directed into D_+. It is well known that small amplitude acoustic waves with harmonic time dependence of the form $e^{i\omega t}$, where ω is the angular frequency, obey the Helmholtz or reduced wave equation [5]

$$\nabla^2\phi + k^2\phi = 0 \tag{1}$$

where $\phi(p)$ is the excess acoustic pressure at a point p in the fluid or on the surface. Here k is the acoustic wavenumber given by $k = \frac{\omega}{c}$ where c is the speed of sound in the fluid.

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A physical requirement of the problem is that all radiated and scattered waves are outgoing at infinity. This condition can be expressed mathematically as the Sommerfeld radiation condition [3,4,5]

$$\lim_{|\underline{p}|=r\to\infty} r\left\{\frac{\partial\phi}{\partial r}(\underline{p}) - ik\phi(\underline{p})\right\} = 0 \tag{2}$$

uniformly in all directions $\frac{\underline{p}}{r}$. If the normal velocity V of the surface is known, then the normal derivative of the acoustic pressure is given by

$$\frac{\partial\phi}{\partial n}(p) = -i\omega\rho_f V(p) \tag{3}$$

where ρ_f is the density of the fluid. Further, it can be shown that equation (1) along with boundary conditions (2) and (3) has a unique solution for all positive values of the wavenumber k.

However, it is clear that it is not feasible to use a domain technique, such as finite elements, because of the infinite domain of (1). For this reason, most workers choose to reformulate (1) as an integral equation over the structure surface S. This has the immediate advantage of reducing the domain from the three-dimensional infinite region D_+ to the two-dimensional finite surface S, and that the radiation condition (2) is automatically satisfied, see [3,4,5]. Here we have chosen to work with a direct integral equation using Green's second theorem. An alternative indirect formulation could have been obtained using a layer potential appropach, see [4,5,7].

Using Green's second theorem it is possible to show that

$$\int_S(\phi(p)\frac{\partial G_k}{\partial n_q}(p,q) - G_k(p,q)\frac{\partial\phi}{\partial n_q}(q))dS_q = \begin{cases} -\phi_{inc}(p) & p\in D_- \\ C(p)\phi(p)-\phi_{inc}(p) & p\in S \\ \phi(p)-\phi_{inc}(p) & p\in D_+ \end{cases} \tag{4}$$

where

$$G_k(p,q) = \frac{e^{ik|p-q|}}{4\pi\mid p-q\mid} \tag{5}$$

is the free-space Green's function for Helmholtz equation and $\phi_{inc}(p)$ is due to any incident wave which may be present. The function $C(p)$ is calculated from the solid angle subtendend by the structure surface at p. For a point on a smooth surface, that is a point where the tangent plane to S is unique, it can be shown that $C(p) = \frac{1}{2}$.

We can write equation (4) for p on a smooth portion of S in operator notation as

$$(-\tfrac{1}{2}I + M_k)\phi = L_k\frac{\partial\phi}{\partial n} - \phi_{inc} \tag{6}$$

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where

$$(L_k\sigma)(p) = \int_S \sigma(q)G_k(p,q)dS_q \tag{7}$$

and

$$(M_k\sigma)(p) = \int_S \sigma(q)\frac{\partial G_k}{\partial n_q}(p,q)dS_q \tag{8}$$

are the single and double layer potential operators respectively.

It can be shown, however, that for a countable set of values of k, denoted I_S, the operators $(-\frac{1}{2}I + M_k)$ and L_k are singular and so (6) does not have a unique solution [3]. This problem is only due to the integral equation formulation employed, and is not associated with any physical effect. Over the past twenty-five or so years a number of methods have been proposed for overcoming this problem, see [2,9,10,11,12,14,15]. Here we shall employ the method due to Burton and Miller [2].

By differentiating (6) with respect to $\underline{n}_p$, the normal at p, we obtain

$$N_k\phi = (\tfrac{1}{2}I + M_k^T)\frac{\partial\phi}{\partial n} + \frac{\partial\phi_{inc}}{\partial n} \tag{9}$$

where

$$M_k^T\sigma(p) = \int_S \sigma(q)\frac{\partial G_k}{\partial n_p}dS_q = \frac{\partial}{\partial n_p}(L_k\sigma)(p) \tag{10}$$

and

$$N_k\sigma(p) = \frac{\partial}{\partial n_p}\int_S \sigma(q)\frac{\partial G_k}{\partial n_q}(p,q)dS_q = \frac{\partial}{\partial n_p}(M_k\sigma)(p). \tag{11}$$

Strictly speaking, the differentiation with respect to $\underline{n}_p$ cannot be taken inside the integral sign in (11) since the resulting kernel function would have a non-integrable singularity. However, this may be done in practice provided the integral is interpreted in the sense of a finite part integral [6].

The Burton and Miller formulation consists of taking a linear combination of (6) and (9) in the form

$$(-\tfrac{1}{2}I + M_k + \alpha N_k)\phi = [L_k + \alpha(\tfrac{1}{2}I + M_k^T)]\frac{\partial\phi}{\partial n} + (\phi_{inc} + \frac{\partial\phi_{inc}}{\partial n}) \tag{12}$$

where α is a complex-valued coupling parameter. It can be shown that provided $Im(\alpha) >$ 0 then (12) will have a unique solution for all real k [2]. Further, it can be shown that the conditioning of (12), and hence the accuracy of a numerical discretisation of (12), depends on the choice of α. Recent results have show that the almost optimal choice of α for obtaining a well conditioned formulation is $\alpha = \frac{i}{k}$ [1,8,7].

Equation (12) can be solved for ϕ numerically using a piecewise constant collocation method. The surface S is divided up into n elements S_j, $j = 1, \ldots, n$ and we choose

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a point $p_j \in S_j$, usually the centroid, to be a collocation point. The unknown pressure ϕ and the normal surface velocity V are now approximated by the piecewise constant functions

$$\begin{aligned} \phi_n(p) &= \textstyle\sum_{i=1}^{n} \phi_i \psi_i(p) \\ V_n(p) &= \textstyle\sum_{i=1}^{n} V_i \psi_i(p) \end{aligned} \tag{13}$$

where

$$\psi_i(p) = \begin{cases} 1 & \text{if } p \in S_i \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

and $\phi_i \approx \phi(p_1)$, $\phi_2 \approx \phi(p_2)$ and so on. Substituting (13) into (12) for each value of p_j yields the matrix equation

$$A\underline{\phi} = i\omega\rho_f B\underline{V} + \underline{c} \tag{15}$$

where $\underline{\phi} = [\phi_1, \phi_2, \ldots, \phi_n]^T$, $\underline{V} = [V_1, V_2, \ldots, V_n]^T$ and $\underline{c} = [\phi_{inc}(p_1) + \alpha\frac{\partial\phi_{inc}}{\partial n}(p_1), \ldots, \phi_{inc}(p_n) + \alpha\frac{\partial\phi_{inc}}{\partial n}(p_n)]^T$, and the matrices A and B are given by the appropriate discretisation of the integral operators.

Clearly we need to take care when evaluating the diagonal elements of A and B since the Green's function and its first derivatives have weak inverse distance singularities. However, the second derivative of the Green's function has a $\frac{1}{|p-q|^3}$ singularity, and we need some further analysis before we can evaluate it. Using a result due to Meyer et al [10] we can write

$$\int_S \frac{\partial^2 G_k}{\partial n_p \partial n_q}\sigma(q)dS_q = \int_S (\sigma(q) - \sigma(p))\frac{\partial^2 G_k}{\partial n_p \partial n_q}dS_q + \sigma(p)k^2 \int_S G_k(p,q)\underline{n}_p.\underline{n}_q dS_q. \tag{16}$$

With our choice of basis functions, when p and q are in the same element the first integral on the right hand side of (16) is zero whilst the second integal is only weakly singular and can be evaluated by an appropriate quadrature rule.

We shall now consider the equations of motion of the elastic structure D_-. The finite element method is a well known technique for analysing the motion of an elastic structure. Assuming harmonic time dependence of the form $e^{i\omega t}$ and in the absence of structural damping, the finite element equations for the strucural analysis can be written as [17]

$$(K - \omega^2 M)\underline{q} = \underline{f}^{(k)} + \underline{f}^{(i)} \tag{17}$$

where K and M are the stiffness and mass matrices; $\underline{q}$ is the vector of nodal displacements and $\underline{f}^{(k)}$ and $\underline{f}^{(i)}$ are consistent load vectors due to known applied forces and fluid-structure interaction forces respectively.

It can be shown that the interaction forces are related to the acoustic pressure through

$$\underline{f}^{(i)} = -L\underline{\phi} \tag{18}$$

where L is a coupling matrix, details of which are given in Wilton [16] or Harris [7]. Thus we can write

$$\underline{q} = (K - \omega^2 M)^{-1}(\underline{f}^{(k)} - L\underline{\phi}). \tag{19}$$

Further, the normal particle velocity $\underline{V}$ at the fluid nodes is related to $\underline{q}$ through

$$\underline{V} = -i\omega L'\underline{q} \tag{20}$$

where the elements of L' are simply the components of the outward normal at the fluid nodes [7,16]. Substituting (19) and (20) into (15) yields

$$(A + DL)\underline{\phi} = D\underline{f}^{(k)} + \underline{c} \tag{21}$$

where $D = \omega^2 \rho_f B L'(K - \omega^2 M)^{-1}$. Once the pressure is known on the surface, we can use (19) and (20) to find the surface velocity, and use (4) with $p \in D_+$ to find the acoustic pressure at any point in the exterior fluid.

For a piezoelectric sonar transducer, the consistant load vector is of the form

$$\underline{f}^{(k)} = \int_V B^T \underline{\sigma}_0 dV \tag{22}$$

where B is the elastic strain-displacement matrix [17], and $\underline{\sigma}_0$ is the stress due to the piezoelectric effect. This is related to the electric field strength $\underline{E}$ through [13]

$$\underline{\sigma}_0 = e_p \underline{E} \tag{23}$$

where e_p is the tensor of piezoelectric parameters. For the simple ring transducers under consideration here, $\underline{\sigma}_0$ has only one non-zero component in the radial direction, which is constant. Although the magnitude of $\underline{\sigma}_0$ is not known, we can assume an arbitrary value and use this to determine how the shape of the response function changes with frequency.

3 EXPERIMENTAL DETERMINATION OF THE RESPONSE

In order to validate our numerical method for accurately predicting the frequencies at which the maximum response in a sonar transducer occurs, we make a comparison between some computed and experimental results. The experimental results are for three ring type transducers. These consist of a ring of ceramic material which is ractangular in cross-section and which may be surrounded by a ring of metal such as aluminium. For the transducers considered here, the inner radius of the ring is 50.8 mm, the height 28mm and the thickness of the ceramic material is 6.35 mm. Transducer A does not

Transducer	Number of Finite Elements	Number of Boundary Elements
A	40	24
B	60	26
C	80	28

Table 1: The number of finite elements and boundary elements used to model each transducer.

have an outer ring of metal, whilst transducers B and C have outer rings of aluminium of thickness 1.5875 mm and 4.7625 mm respectively.

As an initial guess to the resonant frequency in water, we determine the resonant frequency in air by performing a loop test on the transducer as follows. A known voltage is applied to the transducer at different frequencies and the impedance of the transducer is measured at each frequency. The frequency which minimises the impedance is the resonant frequency of the transducer. The resonant frequency in air is now used as a starting point for finding the resonant frequency in water.

Each transducer was immersed in water in a test tank which was approximately 2.3 metres wide by 5.25 metres long and 2 metres deep. The transducer was excited by a unit alternating voltage applied across the electrodes at different frequencies, and the acoustic pressure was measured one metre from the transducer. Clearly this situation was not ideal since there was a strong possibility of reflections from the sides and the bottom of the tank and, to a lesser extent, from the surface of the water, although sides of the tank were covered in a material designed to minimise any reflections. To further complicate the situation, the transducer had to be put in a bag of castor oil, since the water would short circuit the terminals of the ceramic ring. Castor oil was used since it has almost identical acoustic properties to water but is a poor conductor. We have assumed that any effects from the bag, on the acoustic field, were negligible.

This set-up is likely to introduce a number of sources of experimental error into the results that we obtain. To minimise the effect of these errors, the experiment was repeated a number of times in different positions in the tank, and the results averaged.

4 Results and Conclusions

The numbers of quadratic axisymmetric finite elements and linear axisymmetric boundary elements used to model each transducer are shown in Table 1.

To find the peak response the problem must be solved for a number of different

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Transducer	Computed Natural Frequency	Experimental Resonant Frequency in Air
A	9634	9755
B	10058	10225
C	10593	10770

Table 2: The appropriate natural frequency, in Hz, of the sonar transducers considered.

frequencies. However this is expensive since the boundary element matrices have to be re-computed for each new frequency. We can obtain an initial guess for the frequency which gives the peak response by finding the natural frequency of the structure, in a vacuum, which has an eigenvector, or mode shape, similar to the displacements which we are inducing in the transducer, and hopefully the frequency giving the peak response is close to this natural frequency. Table 2 gives the appropriate computed natural frequency in vacuo (in hertz) for each type of transducer with the experimental resonant frequency in air. There is close agreement between the computed and experimental natural frequency, which gives us some confidence in our measuring instruments. Since all three transducers have a natural frequency at about 10 KHz, we shall study the response in the frequency range 5-15 KHz. We note that it is possible to use these high frequencies, and hence high wavenumbers, since the dimensions of the transducers are relatively small. For example, using the largest transducer, C, the maximum value of $d = | p - q |$ is 0.12695m. For $f = 15$ KHz we have $\omega = 94247.78$ and $k = 62.8319$, and hence the maximum value of kd is 7.9766.

The results presented here are for the absolute value of the acoustic pressure at one metre from the transducer in both the radial and the axial directions. The numerical results have been scaled to give pressures of the same magnitude as the experimental results. This is allowed since we have chosen the value of $\underline{\sigma}_r$ arbitrarily.

From our numerical results we were able to determine that the peak response for each transducer lies in the range 7-8.5 KHz. Figures 1, 2 and 3 show the acoustic pressure one metre in the radial direction from transdueucer A, B and C respectively over this frequency range. It can be seen that there is a good agreement between the computed results and the experimental results for transducers A and B. The results for transducer C do not agree so well, but the measured pressure for transducer C is smaller than that for A and B and so could be more susceptible to experimental error due to reflections and other spurious acoustic waves in the tank. Similar results were obtained for the acoustic pressure one meter from each transducer in the axial direction.

It is clear that this numerical technique can be used to predict the response patterns of sonar transducers. If accurate data is available on the piezoelectric properties of the ceramic part, then it is feasible to extend the finite element method to obtain an accurate model to predict the exact response [13].

On the experimental side, it would seem that we need to obtain more accurate results when trying to measure smaller pressures. However, this would probably require the use of a bigger test tank to reduce the interference from any echoes.

References

[1] Amini S (1989). On the choice of the coupling parameter in boundary integral formulations of the exterior acoustic problem. *Appl. Anal.* 35, pp 75-92.

[2] Burton A J and Miller G F (1971). The application of integral equation methods for the numerical solution of boundary value problems. *Proc. Roy. Soc. London,* A232, pp 201-210.

[3] Burton A J (1973). The solution of Helmholtz equation in exterior domains using integral equations. *NPL report NAC30, National Physical Laboratory,* Teddington, Middlesex, UK.

[4] Burton A J (1976). Numerical solution of acoustic radiation problems. *NPL report NAC 92/615/OC, National Physical Laboratoty,* Teddington, Middlesex, UK.

[5] Colton D and Kress R (1983). *Integral equation methods in scattering theory.* Wiley, NY.

[6] Hadamard J L L D (1923). *Lectures on Cauchy's problem in partial differential equations.* Yale University Press.

[7] Harris PJ (1990). *The numerical solution of the dynamic fluid-structure interaction problem.* Ph.D. thesis, Polytechnic South West, Plymouth, Devon, UK.

[8] Harris P J and Amini S (1990). On the Burton and Miller boundary integral formulation of the exterior acoustic problem. *Preprint 17/90,* School of maths and stat, The Universtiy of Birmingham, Birmingham, UK. Also, *Technical report MCS-90-05,* The University of Salford, Salford, UK. Submitted for publication.

[9] Jones D S (1974). Integral equations for the exterior acoustic problem. *Quart. J. Mech. Appl. Math.* Vol XXVII, pt 1.

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[10] Meyer W L, Bell W A, Zinn B T and Stallybrass H P (1978). Boundary integral solution of three-dimensional acoustic radiation problems. *J. Sound and Vibration* 59(2) pp 245-262.

[11] Panich O P (1965). On the question of solvability of the exterior boundary value problems for the wave equation and Maxwell's equation. *Russian Math. Surv.* 20, pp 221-226.

[12] Schenck H A (1968). Improved integral formulations for acoustic radiation problems. *J. Acoust. Soc. of Am.* 44, pp 41-58.

[13] Smith B V (1988). Finite element principles. *Proc. I.O.A.* Vol 10, pt 9; pp 16-37.

[14] Terai T (1980). On the calculation of sound fields around three-dimensional objects by integral equation methods. *J. Sound. and Vibration,* 69(1), pp 71-100.

[15] Ursell F (1973). On the exterior problems of acoustics. *Math. Proc. Camb. Phil. Soc.* 74, pp 117-125.

[16] Wilton D T (1978). Acoustic radiation and scattering from elastic structures. *Int. J. Numer. Methods. Engng.* 13, pp 123-138.

[17] Zienkiewicz O C (1971). *The finite element method in engineering science.* Mc.Graw-Hill, London.

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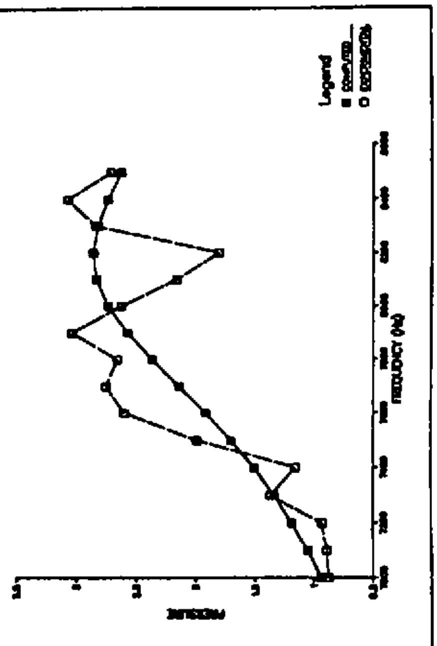

FIGURE 3. The acoustic pressure 1m from transducer C in the radial direction for the frequency range 7000 Hz to 8500 Hz.

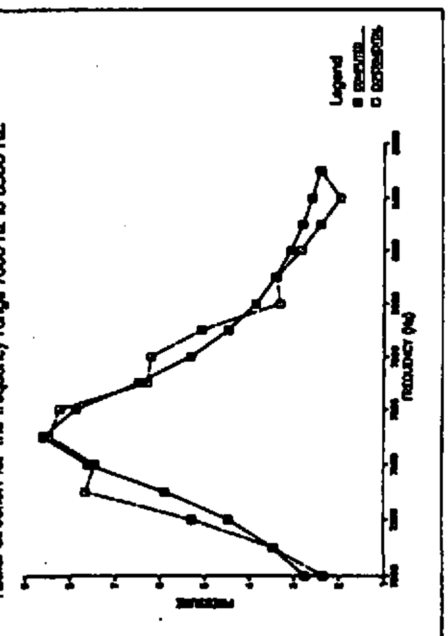

FIGURE 1. The acoustic pressure 1m from transducer A in the radial direction for the frequency range 7000 Hz to 8500 Hz.

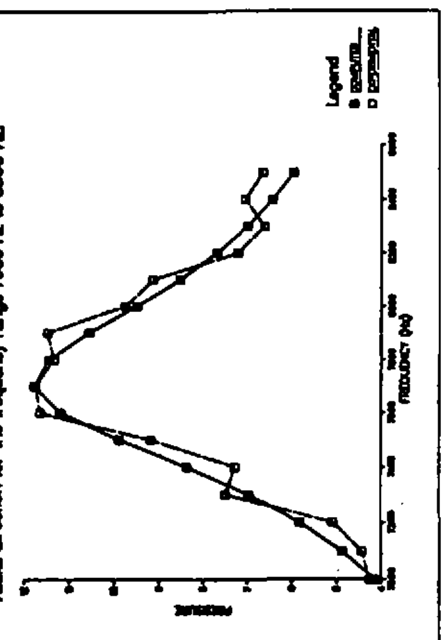

FIGURE 2. The acoustic pressure 1m from transducer B in the radial direction for the frequency range 7000 Hz to 8500 Hz.