TRANSDUCER TOLERANCE EFFECTS ON CONVENTIONAL AND MAXIMUM-LIKELIHOOD BEAMFORMERS

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INTRODUCTION

A frequently occurring problem in sonar engineering is to assess the likely degradation in performance suffered by a system when its various subcomponents depart from theoretically ideal behaviour. At the 'wet' end of a sonar system, such departures may be due to finite electrical tolerances in the transducers, or to constructional inaccuracies in the array geometry. Further along a digital system, quantisation and sampling effects might be considered under the same heading. The present paper is concerned only with the effects of phase and amplitude errors in the hydrophones making up a passive array. For conventional beamforming this problem has been treated by many authors (see, eg, [1] and references therein) by modelling the errors as random variables and examining the statistics (usually the mean) of the resulting random beam pattern. The main aim of this paper is to extend the treatment to a narrowband maximum-likelihood (ML) beamformer by considering the statistics of the 'direction function' (DF) (which, for both conventional and ML beamformers is the output power as a function of steer angle) when the array is placed in a specified sound field. By using the reciprocal of the DF in the ML case, the tolerance effects can be analysed for both beamformers in terms of random perturbations of a quadratic form, using results whose derivation is briefly outlined in the Appendix. To provide a comparison between the tolerance effects in the two cases, the more familiar case of the conventional beamformer is discussed first, and a parallel treatment of the ML beamformer follows.

SOUND FIELD MODEL

Suppose that the array consists of N omnidirectional hydrophones in some arbitrary configuration with the j'th element at position $\underline{x}_j$. Only azimuthal variation in source direction, specified by the bearing angle b, will be considered. The incident sound field is assumed to be due to p farfield point sources emitting narrowband complex gaussian signals at the same frequency; the signal from the k'th source at bearing b_k is received with power level s_k at the array. Each transducer also receives a narrowband complex noise signal at the same frequency and with power n; the noise signals at each hydrophone and all the incident plane-wave signals are assumed mutually uncorrelated. Suppose that a signal from direction b has the wave vector $\underline{g}(b)$ and define the direction vector $\underline{a}(b)$ by

$$\underline{a}(b) = (\exp(i\underline{g}(b).\underline{x}_1), \ldots, \exp(i\underline{g}(b).\underline{x}_N))^{tr}$$

and also define $\underline{a}_k = \underline{a}(b_k)$

If M is the matrix whose (i,j)'th element is the covariance between the total signals at the i'th and j'th hydrophones, then the general form of M which will be considered is thus (using + to denote hermitian transpose):

$$M = nI + \sum s_k \underline{a}_k \underline{a}_k^+ \qquad (1)$$

where the sum is taken over the p sources. The basic beamforming operation consists in multiplying the signal from the j'th hydrophone by the complex weight w_j^* (where * denotes complex conjugation) and summing the resulting N signals. If $\underline{w}$ is the weighting vector with j'th component w_j, the beamformer output power is given by $\underline{w}^+M\underline{w}$. Strictly speaking this is an expectation over the random signals, but a sufficiently long integration time (and stationary sound field statistics) will be assumed for $\underline{w}^+M\underline{w}$ to be taken as the observed value.

TRANSDUCER ERROR MODEL

The output of the j'th transducer is assumed subject to a proportional amplitude error e_{aj} and a phase error e_{fj} so that the transducer output signal is the true (in-water) signal multiplied by a factor

$$1 + e_{aj} + ie_{fj} = 1 + d_j$$

The e_a's and e_f's are taken to be zero-mean random variables, all mutually uncorrelated and with common variance q/2. The assumption of equal phase and amplitude variances simplifies the algebra, but the results are not crucially affected if instead the amplitude errors have a common variance $\mathrm{var}(e_a)$ and the phase errors a common variance $\mathrm{var}(e_f)$; the size of the errors is still expressed by the single parameter $q = \mathrm{var}(e_a) + \mathrm{var}(e_f)$. A further simplifying assumption which will be made is that the e_a's and e_f's are all gaussian. The assumption of independence of errors between different transducers is realistic for the electrical errors considered here. In the treatment of other kinds of error, for example hydrophone position errors, the assumption will not generally be realistic and a more involved analysis is necessary.

Note that the random errors may be regarded either as fixed for a particular realisation of the array, in which case the statistical ensemble of DF's arises from a corresponding ensemble of hardware realisations, or alternatively as slowly time-varying quantities (compared to the integration time), when the DF itself becomes randomly time-varying.

THE CONVENTIONAL BEAMFORMER

Conventional beamforming is achieved by setting $\underline{w} = \underline{a}(b)$ for a chosen look direction b. (For simplicity, uniform amplitude shading will be assumed). The DF is therefore

$$P_o(b) = \underline{a}^+(b)M\underline{a}(b)$$

The transducer errors have the effect of replacing the weight vector $\underline{w}$ by $(1 + D)\underline{w}$ where $D = \mathrm{diag}(d_1, \ldots, d_N)$. The perturbed DF is thus

$$P(b) = \underline{a}^+(b)(1 + D)^+M(1 + D)\underline{a}(b)$$

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From the results in the Appendix, the mean and variance of P(b) are given by

$$EP(b) = P_o(b) + q.tr(M)$$

$$var\ P(b) = 2q\underline{a}^+(b)M^2\underline{a}(b) + q^2.tr(M^2) \qquad (2)$$

Considering first the expression for EP(b), we see that the mean DF is obtained from the unperturbed DF by adding the constant term q.tr(M), which from (1) is qN times the total incident acoustic power. This term therefore plays the role of an omnidirectional statistical 'background level'. As might be expected, this background level increases with the error parameter q. It is obvious that the relative effect of the background term is least where $P_o(b)$ is large, and only begins to become important at bearings b where $P_o(b) \sim q.tr(M)$. For $P_o(b) \ll q.tr(M)$ it is evident that the mean DF flattens out at a level dependent on q, and is almost entirely unaffected by the shape of the unperturbed DF.

Knowledge of the mean DF is useful only if the likely variation of individual realisations about this mean can be estimated, and computation of var P(b) serves this purpose. Again we have a sum of a directional and a nondirectional term, but now both increase with the error parameter q. The variance expression takes a particularly simple form in the case of a single source in zero noise; thus we take p = 1 and n = 0 in (1), giving $M = s_1\underline{a}_1\underline{a}_1^+$. It is easy to verify that equations (2) then become

$$EP(b) = P_o(b) + qNs_1$$

$$var\ P(b) = 2qNs_1P_o(b) + (qNs_1)^2 \qquad (3)$$

The mean and variance are thus both expressed in terms of the unperturbed DF $P_o(b)$ and the constant background term qNs_1. The behaviour of the variance resembles that of the mean, with an omnidirectional term added to a multiple of $P_o(b)$; near peaks in $P_o(b)$ the variance is approximately proportional to $P_o(b)$, and when $P_o(b)$ is small it flattens out to a constant value. The peak value of $P_o(b)$ occurs at $b = b_1$ when $P_o(b_1) = N^2s_1$. The background level is thus in this case lower than the peak by a factor q/N (independently of the source strength s_1), reflecting the fact that tolerance effects on the conventional beamformer become less serious as the number of transducers is increased.

In this simple case, it is interesting to note that an alternative derivation (see [2]) of the formulae for mean and variance proceeds by noting that the real and imaginary parts of the output signal are uncorrelated gaussian variates; the array output power then follows a noncentral chi-square distribution and the given formulae are immediate applications of its standard properties.

For a more general M given by (1) with n > 0 and p > 1, the mean and variance of the DF can no longer both be expressed simply as sums and products of the unperturbed DF and the background level. The form of equations (2) suggests a principal axis transformation of the quadratic forms, which results in the expression of the DF as a sum of 'eigenvector beams', and this is done below for the ML beamformer. Here it will only be noted that in the formula for EP(b), if the discrete sources are sufficiently spaced in bearing and of high enough S/N, the unperturbed DF around b_j is approximately the same as if only source j were present, but the omnidirectional term is increased by contributions from all the other sources, as their energy leaks in via the degraded sidelobe response of the beamformer. This does not lead to the conclusion that tolerances of a fixed size have a more serious effect on performance when many sources are present, since in this case the unperturbed DF will already be presenting a somewhat uninformative picture, with incomplete resolution of all the sources.

Figure 1 shows the unperturbed DF for a 16-element line array with half-wavelength spacing with sources of strength 20 dB, 9.5 dB and 0 dB at bearings of -40, -34 and 30 degrees from broadside respectively, in unit noise. If the amplitude and phase errors have standard deviations of 20% and 11.5 degrees respectively (q = 0.08), the background level turns out to be 21.5 dB. The customary use of a dB scale makes the addition 'by eye' of the unperturbed DF and the omnidirectional background level to give the mean particularly convenient. The middle curve in Figure 2 shows the resulting mean DF, together with bounds of plus and minus one standard deviation. It is clear that the DF in the region of the two strong unresolved sources will be only slightly degraded, with some variability affecting the sidelobe structure, but in other directions the DF is highly variable, and the weak source is likely to be lost.

THE ML BEAMFORMER

The so-called ML beamforming algorithm (see, eg, [3]) derives from the minimisation problem :

$$\text{minimise } \underline{w}^{+}M\underline{w} \quad \text{subject to } \underline{w}^{+}\underline{a}(b) = 1,$$

ie, for a chosen look direction b, minimise the beamformer output power subject to the constraint that the beamformer is 'transparent' (has unity gain) to signals from that direction. The well-known solution to the problem, for nonsingular M, is given by

$$\underline{w} = M^{-1}\underline{a}(b)/R_o(b)$$

where R_o is the reciprocal of the beamformer output power (and therefore its negative in dB) and is given by

$$R_o(b) = \underline{a}^{+}(b)M^{-1}\underline{a}(b)$$

Using the reciprocal output power enables the results of the Appendix to be applied, but with the difference that since the beamformer computes its weight vector in a data-adaptive fashion, the effect of the errors cannot be analysed as for the conventional case simply by replacing $\underline{w}$ by $(1 + D)\underline{w}$. Instead we note that what the adaptive beamformer does is to solve the problem

$$\text{minimise } \underline{w}^{+}(1 + D)^{+}M(1 + D)\underline{w} \quad \text{subject to } \underline{w}^{+}\underline{a}(b) = 1$$

since the constraint is assumed to be correctly implemented and the only errors are in the perturbed version of the in-water covariance which the beamformer 'sees'. It follows that

$$R(b) = \underline{a}^+(b)((1 + D)^+M(1 + D))^{-1}\underline{a}(b)$$

$$= \underline{a}^+(b)(1 - D)M^{-1}(1 - D)^+\underline{a}(b) \text{ to 1st order in D.}$$

The Appendix now applies (it can be checked that the replacement of D by $-D^+$ makes no difference) and for the mean and variance of R(b) we have

$$ER(b) = R_o(b) + q.tr(M^{-1})$$

$$\text{var } R(b) = 2q\,.\,\underline{a}^+(b)M^{-2}\underline{a}(b) + q^2tr(M^{-2}) \qquad (4)$$

In the first of these equations, an 'omnidirectional background level' effect occurs, exactly analogous to the conventional case, but the background term is no longer simply a multiple of the total acoustic power. The eigenvector beam decomposition of R_o takes the form

$$R_o(b) = \sum u_k f_k(b)$$

with the sum taken over the N eigenvalues of M^{-1} which in ascending size are $u_1, \ldots, u_N$, and $f_k(b) = |\underline{a}^+(b)\underline{v}_k|^2$ where $\underline{v}_k$ is the normalised eigenvector corresponding to u_k. The eigenvector beams are thus normalised to a peak value of N. Equations (4) can now be written

$$ER(b) = \sum u_k f_k(b) + q\sum u_k$$

$$\text{var } R(b) = 2q\sum u_k^2 f_k(b) + q^2\sum u_k^2 \qquad (5)$$

with the sums taken over k from 1 to N. Comparison of these equations with the single-source conventional beamformer results in (3) shows that the RHS's of the above equations are the sums of such expressions for sources of strength proportional to u_k.

If we first consider the behaviour of ER(b) in the simplest case, obtained by putting $p = 1$ and $n = 1$ in (1) (n cannot now be set to zero or $R_o(b)$ will become unbounded), then $u_1 = 1/(Ns_1 + 1)$ and all the other eigenvalues are 1. In the look direction $R_o(b_1) = N/(Ns_1 + 1) \simeq 1/s_1$ for high enough S/N, and in directions away from the look direction $R_o(b) \simeq N$, giving a flat 'noise ceiling' dropping to a null in the source direction, whose depth increases with source strength. Ignoring the smallest eigenvalue, the background level is $q(N-1) \sim qN$. The main difference from the conventional analysis is now apparent, since the background level is fixed relative to the noise ceiling and not the DF value (peak or null) in the source direction. Thus the behaviour of ER(b) suggests that if the source is weak enough, the DF is not significantly affected by the transducer errors; as the source is made stronger, a 'saturation' effect comes into play with the depth of the null limited at the background level.

A qualitative idea of what will happen in the case of several sources can be gained by considering, as before, p sources ($1 < p < N$) whose direction vectors are approximately orthogonal, so that the first p eigenvalues approximate to the small value $u_k = 1/(Ns_k +1)$ and the remaining N-p take the value 1. In the expression for ER(b) the constant background term is thus approximately q(N-p), showing that the presence of a greater number of sufficiently strong sources lowers the background level so that the deep nulls are less severely limited. This decrease in the background level becomes progressively less marked as more sources are added since they soon fail to be even approximately orthogonal and not all the p eigenvalues will be small.

For the same sound field as in Figures 1 and 2, Figure 3 shows the unperturbed inverse DF $R_o(b)$, with the ML beamformer successfully resolving the three sources and correctly estimating their respective strengths. The background level for the same errors (q = 0.08) is now found to be 0.2 dB and the resulting mean plus and minus one standard deviation appears in Figure 4. Nearly all the information regarding the relative strengths of the sources is lost, with the two strong sources appearing at approximately the same strength as the weakest. Moreover, the figure suggests that the beamformer will only just be able to resolve these two strong sources.

When a further 12 sources, all of strength -6 dB, are added at 10 degree intervals from -25 to 85 degrees, Figure 3 is replaced by Figure 5, and Figure 4 by Figure 6. Because of the new eigenvalue distribution caused by the extra sources, with a larger proportion of small eigenvalues, the background level drops to -4 dB, and as a result the DF does provide a slightly better indication of where the strongest sources are. However, in this example the extra sources have the effect of causing the two strong sources in the unperturbed DF to be already somewhat less well resolved, in the sense that the peak between the two source nulls is lower, and so the lowered background level does not yield any noticeable improvement.

Some simulation results corresponding to these two source distributions are shown in Figures 7 and 8, which correspond to Figures 4 and 6 respectively; four different realisations of the random errors with q = 0.08 give rise to the four DF's in each case. These figures show that the statistical bounds shown in Figures 4 and 6 do provide a useful estimate of how an individual realisation is likely to behave. It is also interesting to note that as the curves in Figures 4 and 6 succeed in just resolving the two strong sources, so do all four of the realisations.

SUMMARY

An analysis of transducer tolerance effects on a conventional beamformer, using the useful concept of the 'statistical background level', has been outlined, yielding results familiar to sonar engineers. By considering the reciprocal of the array output power, an exactly analogous method has been applied to a ML adaptive beamformer and some differences between the effects in the two cases noted. In particular, the tolerances give rise to a limiting effect in the ML direction function so that the strength of strong sources is underestimated.

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REFERENCES

[1] D.J.Ramsdale and R.A.Howerton, 'Effect of element failures and random errors in amplitude and phase on the sidelobe level attainable with a linear array', JASA 68(3), 901-906, (1980).
[2] J.L.Allen, 'Some Extensions of the Theory of Random Error Effects on Array Patterns', MIT Lincoln Lab Tech. Report No.236 (1961).
[3] S.Haykin (ed), 'Nonlinear Methods of Spectral Analysis', Springer, Berlin (2nd ed 1983).

APPENDIX

This appendix briefly indicates the derivation of the formulae (2) and (4) for mean and variance of the randomly perturbed quadratic form. In the same notation as the main text, and assuming D is zero-mean,

$$EP = \underline{a}^{+}M\underline{a} + E\underline{a}^{+}D^{+}MD\underline{a}$$
$$= P_o + \sum a_i^{*} r_{ij} m_{ij} a_j \qquad \text{where} \qquad Ed_i^{*} d_j = r_{ij}$$

and the sum is taken over i and j from 1 to N. The added term is another positive semidefinite quadratic form whose kernel is the elementwise or Hadamard product of the covariance matrices of the sensor signals and the complex errors respectively, and will simplify only if at least one of these covariances takes a simple form. In the case considered in this paper, (r_{ij}) is just q times the identity and has the effect of picking out the diagonal elements of M. Since $\underline{a}$ is assumed to be a direction vector all its elements have unit modulus and the constant term is thus seen to be q.tr(M), yielding the first half of equations (2) and, with M replaced by its inverse, the first half of equations (4).

If the errors are correlated so that their covariance matrix is no longer diagonal, then a simple generalisation of the formula for the mean is obtained by performing a spectral decomposition of (r_{ij}) into a sum of dyadics; the background term is then generally direction-dependent.

The variance of P is straightforwardly obtained by computing the expectation of P^2. The expressions encountered in the general case will not be given here; it will just be pointed out that when the amplitude and phase errors have equal covariance matrices, and all amplitude errors are uncorrelated with all phase errors, then terms like $Ed_i d_j$ disappear. The further assumption that the error distributions are all gaussian enables the mean of 4th-order terms in the d's to be written in terms of (r_{ij}), whereas all 3rd-order terms vanish. The result is

$$\text{var } P = 2\sum r_{il} m_{ij} m_{kl} a_i^{*} a_j a_k^{*} a_l + \sum r_{il} r_{kj} m_{ij} m_{kl} a_i^{*} a_j a_k^{*} a_l$$

where the sums are taken over i,j,k and l from 1 to N. It is now easy to check that when (r_{ij}) is q times the identity, we get the second half of equations (2), with the obvious modification for equations (4).

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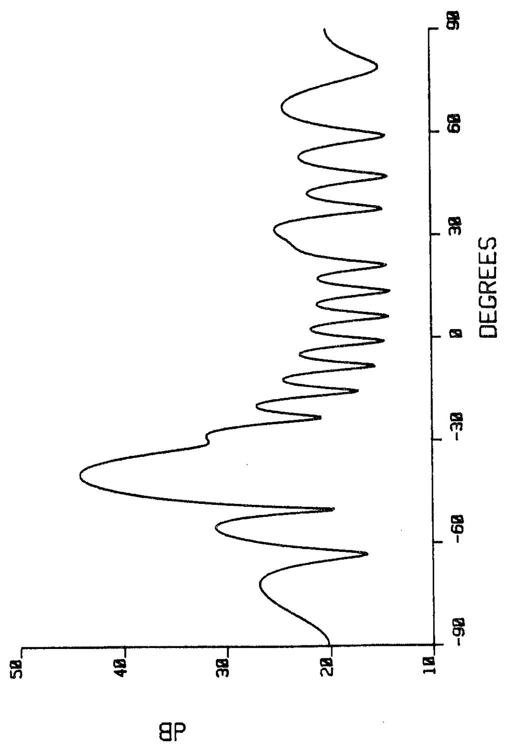

Figure 1 : Conventional, 3 Sources

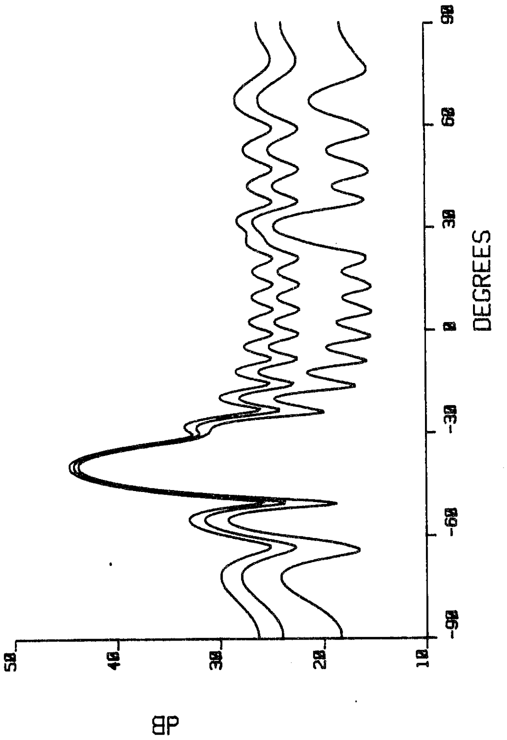

Figure 2 : Conventional, 3 Sources

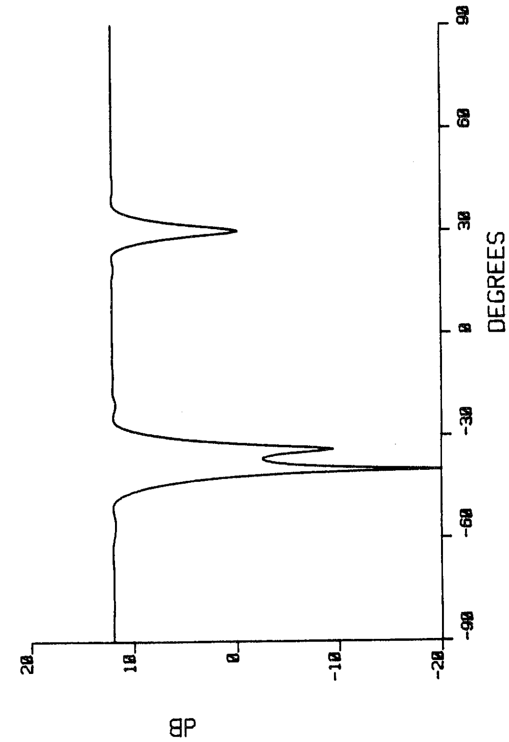

Figure 3 : ML, 3 Sources

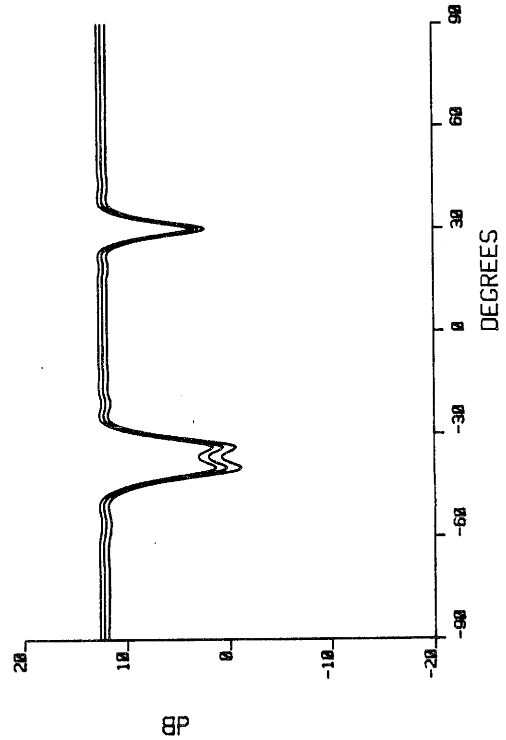

Figure 4 : ML, 3 Sources

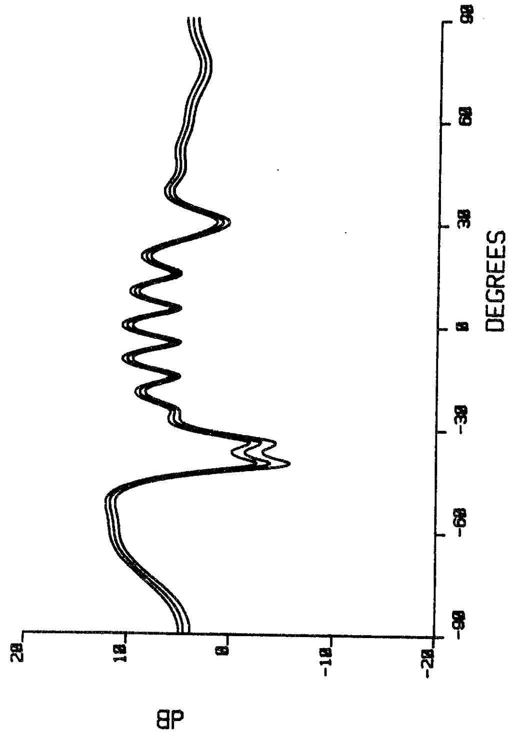

Figure 6 : ML, 15 Sources

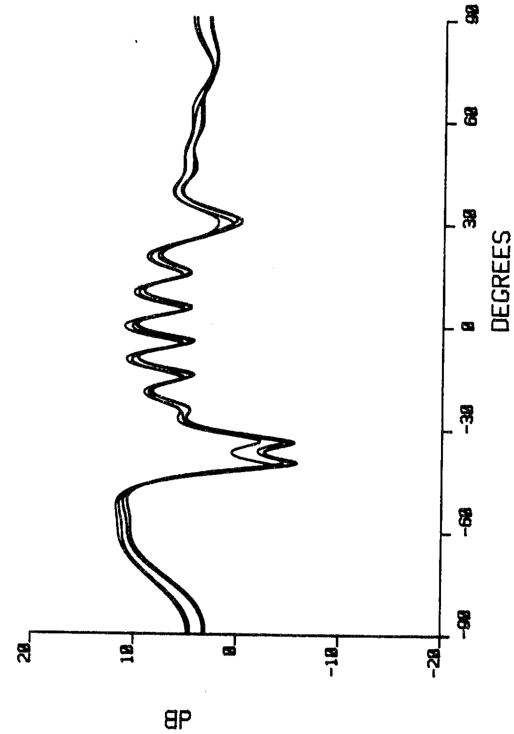

Figure 8 : ML, 15 Sources

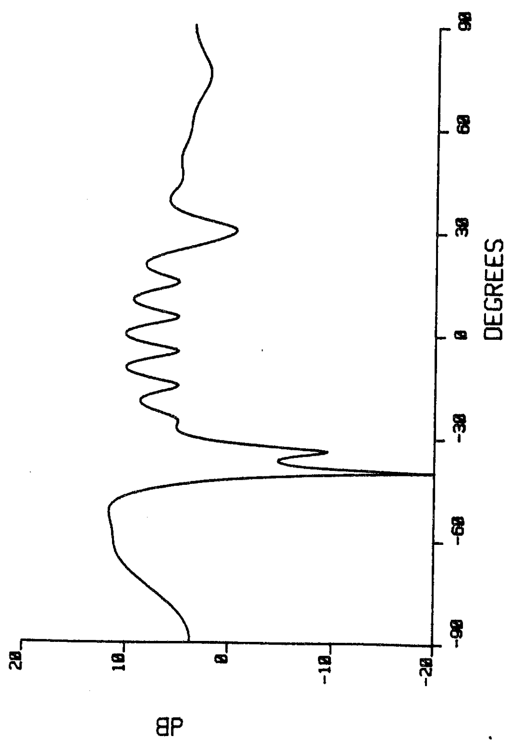

Figure 5 : ML, 15 Sources

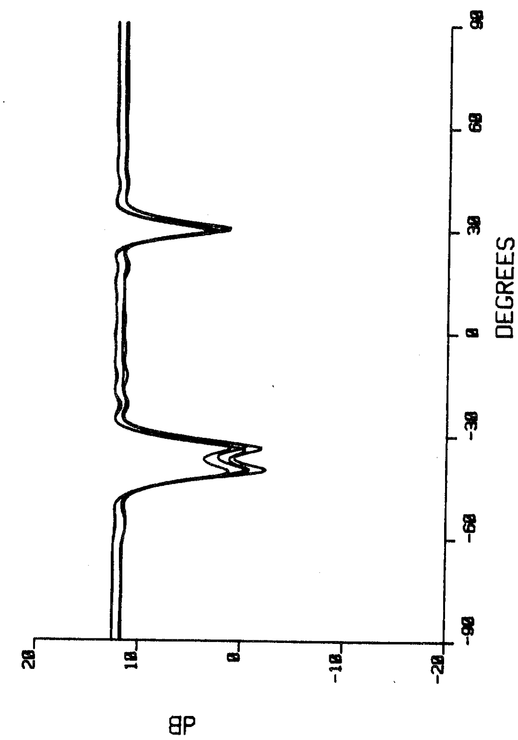

Figure 7 : ML, 3 Sources