The General Theory of Sound Scattered by Sound

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## ABSTRACT

The acoustic stress tensor is derived and its role in the measurement of sound scattered by sound is set forth. A general theory of sound scattered by sound is obtained which is valid for arbitrary configurations of the primary fields. This theory automatically accounts for deviations from ideal planarity of the interacting waves, a feature lacking in previous work. These deviations are shown to contribute nothing to far field scattering.

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## I. INTRODUCTION

In the course of measuring the non-linear interaction of two acoustic waves a special problem arises in the event the microphone is simultaneously irradiated by both waves. It is then necessary to account for the demodulation which occurs on the sensitive element of the receiver. This can in principle be done by integrating the acoustic stress tensor over the active face of the transducer. This tensor is derived in Section II.

Recently the expression for the virtual source strength was demonstrated to be valid for arbitrary configurations of the primary waves.(1) The original derivation(2) tacitly assumed these primary waves to be plane. In spite of this advance the early theory of the scattering of sound by sound which predicted no scattering, has not until now been updated to take into account this new development. This is done in Section III where a very simple conjecture is given of the fact that deviations from planarity of the primary waves, such as are known to occur in the Fresnel zone of radiators, contribute nothing to the scattering process.

## II. STRESS TENSORS

Following the notation of Landau and Lifshitz, The Classical Theory of Field(3) the square of the infinitesimal line element $ds^2$ is related to $g_{ik}$ the metric of four dimensional space as follows

$$ds^2 = - g_{ik}\, dx^i\, dx^k \tag{1}$$

In the above relation the $dx^i$ are the coordinate differentials and it is understood that one sums on repeated indexes i and k from 0 to 3.

The equations of motion for any system in a vanishingly small gravitational field are

$$T^{ik}{}_{,k} = 0, \tag{2}$$

in which $T^{ik}$ stands for the energy momentum tensor density of the system under consideration and , k means ordinary differentiation with respect to the $k^{th}$ coordinate. Thus, choosing

$$T^{ik} = (p + \varepsilon)\, u^i\, u^k + {}^{o}g^{ik}\, p\ , \tag{3}$$

the relation Eq. (2) yields the hydrodynamic equations of an inviscid fluid. To see that this is so for non-relativistic motion in which the ordinary three dimensional flow velocity $v^\alpha$ is negligible compared with c the speed of light, we put $u^\alpha = \frac{v^\alpha}{c}$ and $u^o = -1$. The Greek indices $\alpha$ take on the values 1, 2 or 3 corresponding to the three spatial coordinates $x^1$, $x^2$ and $x^3$. The time coordinate is $x^o = ct$. The metric ${}^{o}g^{ik}$ in the absence of a gravitational field has the only non-vanishing components ${}^{o}g^{\alpha\beta} = \delta_{\alpha\beta}$, and ${}^{o}g^{oo} = -1$. The fluid pressure is p, its density $\varepsilon/c^2 = \rho$.

Inserting Eq. (3) into Eq. (2) and separating the equations into the $\alpha$ space components and o time component yields

$$[(c^{-2}\, p + \rho)\, v^\alpha\, v^\beta]_{,\beta} + p_{,\alpha} + c[(c^{-2}p + \rho)\, v^\alpha]_{,0} = 0, \tag{4}$$

and

$$[(c^{-2}p + \rho)\, v^\beta]_{,\beta} + c[(c^{-2}p + \rho)]_{,0} = 0. \tag{5}$$

In most cases $\rho \gg p/c^2$ thus Eq. (4) becomes

$$\frac{\partial \rho v^\alpha v^\beta}{\partial x^\beta} + \frac{\partial p}{\partial x^\alpha} + \frac{\partial \rho v^\alpha}{\partial t} = 0 \tag{6}$$

the familiar equation for conservation of momentum, while Eq. (5) becomes

$$\frac{\partial \rho v^\beta}{\partial x^\beta} + \frac{\partial \rho}{\partial t} = 0, \tag{7}$$

corresponding to mass conservation.

In a by now familiar way Eq. (6) and Eq. (7) may be combined to yield Lighthill's equation

$$c_o^{\,2}\, \Box\, \rho = - \frac{\partial^2}{\partial x^\alpha \partial x^\beta}\, \tau_L^{\alpha\beta}\ , \tag{8}$$

in which $\tau_L^{\alpha\beta}$ stands for Lighthill's stress tensor given by

$$\tau_L^{\alpha\beta} = \rho v^\alpha v^\beta + (p - \rho c^2)\delta^{\alpha\beta}\ , \tag{9}$$

and $c_o$ is the speed of sound.

We will now obtain yet another stress tensor, the acoustic energy momentum complex of which the space-space components are the flux of acoustic momentum.[4] We begin by asking how much power per unit volume w does a pressure p put into a simple source density of strength q. The answer is obvious

$$w = -pq. \tag{10}$$

Then we ask what force per unit volume $f^\alpha$ does a velocity $v^\alpha$ cause to act upon q which is assumed stationary. A simple momentum balance done by integrating the flux of momentum from Eq. (3) over a closed surface surrounding q reveals that

$$f^\alpha = -\rho_o q v^\alpha . \tag{11}$$

It is convenient to introduce $\phi$ the velocity potential which satisfies the equations

$$v^\alpha = \phi^{,\alpha} , \tag{12}$$

$$p = -\rho_o \dot{\phi} \quad \text{and} \tag{13}$$

$$\Box \phi = q . \tag{14}$$

The two equations (10) and (11) can be written as one with the help of $f^k$, a four vector whose space components are $f^\alpha$ and whose time component is $w/c_o$, thus

$$f^k = -\rho_o q \phi^{,k} \tag{15}$$

A tensor $\tau_A^{ik}$ whose divergence yields the negative of the four force density $f^k$ is the desired acoustic complex. That is we desire

$$\frac{\partial \tau_A^{ik}}{\partial x^k} = -f^i \tag{16}$$

With the help of Eq. (14) and Eq. (15), we may write Eq. (16) as follows

$$\begin{aligned} \tau^{ik}_{A,k} &= \rho_o (\Box \phi) \phi^{,k} \\ &= \rho_o \phi^{,k} \phi^{,\ell}_{,\ell} \end{aligned} \tag{17}$$

It is now a simple matter to verify that the following tensor satisfies Eq. (17),

$$\tau_A^{ik} = \rho_o[\phi^{,i}\,\phi^{,k} - 1/2\; {}^o g^{ik}\,\phi^{,\ell}\,\phi_{,\ell}]\;. \tag{18}$$

In terms of the physical quantities, pressure and velocity, the space-space components of $\tau_A^{ik}$ are

$$\tau^{\alpha\beta} = \rho_o v^\alpha v^\beta - 1/2\;\delta^{\alpha\beta}\,(\rho_o\, v^2 - \frac{p^2}{\rho_o c_o^{\,2}}), \tag{19}$$

the familiar flux of acoustic momentum density.

## III. SOUND SCATTERED BY SOUND

In this section an attempt will be made to show why non-plane waves do not scatter each other. In its present state the demonstration which follows lacks the rigour I would prefer it to have.

We start by investigating the properties of the d'Alembertian of the product of the radius vector $x^\alpha$ and the virtual source strength q .

$$\begin{aligned} \Box(x^\alpha q) &= [x^\alpha q]^{,\ell}_{,\ell} \\ &= [x^\alpha q]^{,\beta}_{,\beta} + [x^\alpha q_{,\alpha}]^{,o}_{,o} \\ &= 2\nabla^2 q + x^\alpha[\Box\, q]_{,\alpha}\;. \end{aligned} \tag{20}$$

By expressing the Laplacian of q in terms of its time derivatives and d'Alembertian, Eq. (20) becomes

$$\ddot{q} = \frac{c_o^{\,2}}{2}\;\Box[x^\alpha q_{,\alpha} + q] - \frac{c_o^{\,2}}{2}\,[x^\alpha\;\Box q]_{,\alpha} \tag{21}$$

In evaluating the asymptotic retarded integral solution for the above source distribution, we may as usual ignore contributions from the d'Alembertian thus

$$\int \frac{q]_{t'}}{r}\;dv = \frac{c}{2}\,n_\alpha \int \frac{x^\alpha[\int q dt]_{t'}}{r}\;dv\;\;. \tag{22}$$

Now $\int q dt$ is proportional to $p_1\,p_2$, the product of the sound pressures of the two primary interacting waves. Furthermore in the absence of real sources which is the case in this development,

$$\frac{1}{2}\ \Box(p_1\ p_2) = (p_1)^{,i}\ (p_2)_{,i}\ , \tag{23}$$

a quantity akin to the Lagrangian density.

Introducing Eq. (23) into the right hand side of Eq. (22) shows the scattered wave to be proportional to

$$n_\alpha \int \frac{x^\alpha[(p_1)^{,i}\ (p_2)_{,i}]_{t'}}{r}\ dv\ . \tag{24}$$

Since the function in the square brackets is well behaved (containing no singularities) we may make the following sequence of approximations to Eq. (24)

$$\int \frac{n_\alpha x^\alpha[(p_1)^{,i}\ (p_2)_{,i}]_{t'}}{r}\ dv =$$

$$\int [(p_1)^{,i}\ (p_2)_{,i}]_{t'}\ dv =$$

$$\int [\ \Box(p_1\ p_2)]_{t'}\ dv = 0. \tag{25}$$

This completes the demonstration. The approximations employed could not be invoked at the level of Eq. (22) because of the complete generality of q. The validity of the procedure here employed is supported by the fact that $(p_1)^{,i}\ (p_2)_{,i}$ is non-vanishing even when both primary waves are plane, yet the original theory of sound scattered by sound predicted no scattering for this case.

## IV. AN APOLOGY AND A JUSTIFICATION

I must apologize to my audience and readers for my choice of language in which to cast some of this work, namely, the four-dimensional relativistic notation. I assure you this is not an affectation on my part but stems from the close similarities in the two branches of science, Acoustics and General Relativity.(5) Let me illustrate by an example.

Lighthill's equation may be written in terms of the strain h

$$h = \xi^\alpha_{,\alpha} \tag{26}$$

in which $\xi^{\alpha}$ is the weak field particle displacement. Thus

$$\Box h = \rho_o^{-1} c_o^{-2} \tau^{\alpha\beta}_{L,\alpha,\beta} \tag{27}$$

We see then that an arbitrary stress distribution $\tau_L^{\alpha\beta}$ gives rise to a retarded strain distribution h. A simple calculation shows that a plane h wave has an intensity equal to $\rho_o c_o^3 h^2$ which is evidently proportional to the square of the strain.

The field equations of General Relativity can be written(6) in terms of strains in an otherwise flat space, giving rise to a three space with metric $\gamma_{\alpha\beta}$

$$\Box \gamma_{\alpha\beta} = -\frac{16\pi G}{c^4} \tau_{\alpha\beta} \tag{28}$$

Here G is the gravitational constant and $\tau_{\alpha\beta}$ represents not only stresses of the variety given in Eq. (3) but also the stresses engendered by the gravitational field itself. Thus in General Relativity as in Acoustics, the fields may serve as their own sources.

Finally the energy flux of a plane gravitational wave may be expressed in terms of the fractional change in cross sectional area of an element of surface normal to the wave vector. Thus if $ds^{\hat{}}$ refers to strained space and ds to flat space, the intensity turns out to be(7)

$$-\frac{\pi}{4}\frac{c^5}{G\lambda^2}\;\frac{\langle ds - ds^{\hat{}}\rangle}{ds}\ , \tag{29}$$

in which $\lambda$ is the wavelength of the wave and < > stands for time averaging.

## V. THE SECOND ORDER WAVE EQUATION IN THE PRESENCE OF SOURCES

The properties of the acoustic stress $\tau_A^{ik}$ facilitate the derivation of a wave equation exact to second order and valid in the presence of real sources q.

Combining Eqs. (15) and (16) yields

$$\tau_{A;\,k}^{ik} = \rho_o q\phi^{,i} , \tag{30}$$

which is equivalent to the two equations

$$\tau_{A,\beta}^{\beta\alpha} = -\tau_{A,o}^{o\alpha} + \rho_o q\phi^{,\alpha} \tag{31}$$

and

$$\tau_{A,\alpha}^{o\alpha} = -\tau_{A,o}^{oo} + \rho_o q\phi^{,o}. \tag{32}$$

A second divergence yields

$$\tau_{A,\alpha,\beta}^{\alpha\beta} = -\tau_{A,o,\alpha}^{o\alpha} + (\rho_o q\phi^{,\alpha})_{,\alpha} \tag{34}$$

and

$$\tau_{A,o,\alpha}^{o\alpha} = -\tau_{A,o,o}^{oo} + (\rho_o q\phi^{,o})_{,o}. \tag{35}$$

Combining Eqs. (34) and (35) yields

$$\tau_{A,\alpha,\beta}^{\alpha\beta} = \tau_{A,o,o}^{oo} + (\rho_o q\phi^{,\alpha})_{,\alpha} - (\rho_o q\phi^{,o})_{,o}. \tag{36}$$

Referring to Eqs. (9) and (19) we may now express Lighthills stress $\tau_L^{\alpha\beta}$ in terms of the acoustic stress $\tau_A^{L\beta}$, thus

$$\tau_L^{\alpha L\beta} = \tau_A^{\alpha L\beta} + ( L + p - \rho c^2)\delta^{\alpha\beta} \tag{37}$$

in which L stands for the Lagrangian density

$$L = T - V = 1/2\ \rho_o v^2 - \frac{p^2}{2\rho_o c_o^{\,2}} , \tag{38}$$

and T and V are the kinetic and potential energy densities, respectively.

Next combining Eqs. (36) and (37) there results

$$\tau_{L,\alpha,\beta}^{\alpha\beta} = \tau_{A,o,o}^{oo} + ( L + p - \rho c^2)_{,\alpha}^{,\alpha} + (\rho_o q\phi^{,\alpha})_{,\alpha} - (\rho_o q\phi^{,o})_{,o}. \tag{39}$$

Now $\tau_A^{oo}$ is the acoustic energy density E that is

$$\tau_A^{oo} = E = T + V = 2V + L, \tag{40}$$

which fact combined with the operator relation

$$\nabla^2 = \Box \; _{,o,o} \; , \tag{41}$$

permits Eq. (39) to be written as follows

$$\tau_{L,\alpha,\beta}^{\alpha\beta} = \Box(L + p - \rho c^2) + (2V + 2L + p - \rho c^2)_{,o,o} \tag{42}$$

$$+ (\rho_o q\phi^{,\alpha})_{,\alpha} - (\rho_o q\phi^{,o})_{,o}.$$

The Lagrangian density is in part derivable from a d'Alembertian.

Since

$$L = 1/2 \; \rho_o \phi^{,i} \phi_{,i} \; , \tag{43}$$

We see, using Eq. (14), that

$$L = \Box(1/4 \; \rho_o \phi^2) - 1/2 \; \rho_o q\phi. \tag{44}$$

The Lagragian in Eq. (42) is eliminated using Eq. (44) leading to

$$\tau_{L,\alpha,\beta}^{\alpha\beta} = \Box[L + p - \rho c^2 + 1/2 \; \rho_o \; (\phi^2)_{,o,o}] \tag{45}$$

$$+ (2V + p - \rho c^2)_{,o,o} + (\rho_o q\phi^{,\alpha})_{,\alpha} - (\rho_o q\phi^{,o})_{,o}$$

$$+ (\rho_o q\phi)^{,o}_{,o} \; .$$

This result may be modified by including $\rho q\phi$ within the d'Alembertian

$$\tau_{L,\alpha,\beta}^{\alpha\beta} = \Box[L + p - \rho c^2 + 1/2 \; \rho_o \; (\phi^2)_{,o,o} + \rho_o q\phi]$$

$$+ (2V + p - \rho c^2)_{,o,o} - (\rho_o q^{,\alpha}\phi)_{,\alpha} - (\rho_o q\phi^{,o})_{,o}. \tag{46}$$

Lighthill's equation for arbitrary fluid motion in the presence of sources and correct to second order, is

$$\Box \rho = - \frac{1}{c_o^2} \; \tau_{L,\alpha,\beta}^{\alpha\beta} \; , \tag{47}$$

in which $\tau_{L,\alpha,\beta}^{\alpha\beta}$ can be obtained from Eq. (46). The detailed application of Eq. (47) to particular problems will be deferred for a future paper.

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(7) Footnote 10 of reference 6.