ANALYSIS OF VERTICAL VIBRATION CHARACTERISTICS OF INFINITE PERIODIC SUPPORTED RAIL UNDER TEMPERATURE STRESS BY USING THE FOURIER TRANSFORM METHOD AND THE PERIODIC STRUCTURE THEORY

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While temperature in the rail changes, axial temperature stress will exist. If the axial stress huge enough, it may lead to the rail expansion, excel-out even breakage in continuously welded rail track. Therefore, measuring the temperature stress in the rail accurately is an important concern. As we known, when a beam subject to an axial stress, its dynamic character will be changed. So the vibration test method may be an available method for evaluating the axial stress in the rail. However, the dynamic vibration character of the track under axial stress is not very clear, especially in high frequency range. In this paper, two different track models are established, i.e. the periodic continuously supported single layer beam model and the periodic discretely supported single layer beam model. In both models, the rail is regarded as a Timoshenko beam in order to obtain the dynamic vibration character of the rail in high frequency range and the elastic of the other parts under the rail is simplified as the equivalent spring stiffness. The axial force is simplified as a concentrate force which acts on the center position of the beam section. The dynamic differential equations for each model are solved by using the Fourier transform method with the periodic structure theory. Though the periodic structure theory, the influence of sleepers which are regularly distributed along the track can be taken into consideration. The frequency dispersion curves are obtained through the free vibration of two models. The vertical dynamic displacement amplitudes for different frequency due to a moving harmonic load are also analyzed. The relevant results of two models are compared and it shows that the resonant frequency in high frequency range is more sensitive to the axial stress, so it may be a available index for evaluating the axial stress in the rail.

Keywords: temperature stress, vertical vibration, high frequency, periodic structure, Fourier-transfer method

1. Introduction

How to accurately detect the temperature stress inside the rail of continuously welded rail track (CWR) is a very important issue for the railway maintenance department. Because huge temperature stress may lead to the rail expansion, excel-out even breakage in the CWR.

Currently, a lot of detecting methods have been proposed. These methods can be classified into three categories: destructive method, semi destructive method and non destructive method. The destructive method includes the rail sawing method and drilling method. The semi destructive method includes transverse loading method and observation pile method. Both destructive and semi destructive detection method will destroy the stability of the original track structure, and the detection efficiency is low. In recent years, some non-destructive detection methods have bring into railway engineering for detecting the temperature stress inside the rail, such as Barkhausen noise method, X-ray method and ultrasonic guided wave method. These methods can keep the integrity of the track structure during detection process, but the main problem is that the temperature stress detected by these methods is the stress that distributed on the rail surface or shallow layer and can not reflect the distribution of temperature stress on the whole section of the rail and the accuracy of the results will be affected by the rail defects. Therefore, it is urgent to find a more ideal and accurate non-destructive testing method.

While a beam subjects to an axial force, its vibration characters may be affected, such as the resonant frequency. So it may be a possible method for evaluating the temperature stress inside the rail. Some scholars have carried out research on this aspect and have achieved some results.

Zhu jianyue. et. al. analyze the vibration characters of the track structure by using finite element method [1-3]. The results show that the resonant frequencies of the rail are affected by the axial temperature stress. All resonant frequencies are increase with the increase of the axial tensile stress, and decrease with the increase of the axial compressive stress. The drawback of these researches is that the resonant frequencies in analysis are under 100Hz. In order to improve the sensitivity of the resonance frequency to the axial temperature force, it needs a large sleeper spacing (above 8.4m). Aikawa A[4] established a three-dimensional beam element model of the track structure by using Floquet transform for periodical structure. The results show that the lateral vibration characters are more sensitive to the axial force rather than the vertical vibration characters. However, the Floquet periodical structure theory is complicated and inconvenient to use.

The method which combines the Fourier transform method and the periodic structure theory is also called the wave number finite element method for periodical structure. This method has been used in the analysis of vibration characteristics of track structures. Because the track structure can be regarded as an infinitely long structure along the *x*-direction and sleeper, rail pad and other supporting structure arranged in equal distances under the rail. So this method may be used to analyze the vibration characters of the track structure with axial force. Compared with the Floquet periodical theory, the wave number method for periodical structure is easier, and the analysis process is simpler.

2. Wave number finite element method for periodic structure

For a non supported elastic body, its differential equation of motion can be expressed as[5],

$$[\mathbf{M}]\{\ddot{\mathbf{q}}(x,t)\}+[\mathbf{K}]_0\{\mathbf{q}(x,t)\}+[\mathbf{K}]_1\frac{\partial}{\partial x}\{\mathbf{q}(x,t)\}-[\mathbf{K}]_2\frac{\partial^2}{\partial x^2}\{\mathbf{q}(x,t)\}=\{\mathbf{P}(x,t)\} \tag{1}$$

where $[\mathbf{M}]$, $[\mathbf{K}]_0$, $[\mathbf{K}]_1$, $[\mathbf{K}]_2$ are constant matrix; $\{\mathbf{P}(x,t)\}$ is the load which applied on the rail; $\{\mathbf{q}(x,t)\}$ is the displacement vector of the n nodes on the *x* cross-section, denoted by a 3n vector:

$$\{\mathbf{q}(x,t)\}=(u_1,v_1,w_1,\cdots,u_n,v_n,w_n)^{\mathrm{T}} \tag{2}$$

If attach the supporters which distributed along the track regularly to the elastic body, the load $\{\mathbf{P}(x,t)\}$ can be divided into two parts, i.e.,

$$\{\mathbf{P}(x,t)\}=\{\mathbf{P}_{\mathrm{e}}(x,t)\}+\{\mathbf{P}_{\mathrm{s}}(x,t)\} \tag{3}$$

where $\{\mathbf{P}_{\mathrm{e}}(x,t)\}$ represents the external load which applies on the elastic body, $\{\mathbf{P}_{\mathrm{s}}(x,t)\}$ is the load provided by the supporters. So the displacement vector $\{\mathbf{q}(x,t)\}$ also can be expressed as:

$$\{\mathbf{q}(x,t)\}=\{\mathbf{q}_{\mathrm{e}}(x,t)\}+\{\mathbf{q}_{\mathrm{s}}(x,t)\} \tag{4}$$

ınsert Eq. (4), Eq.(3) into Eq.(1),

$$\begin{cases} [\mathbf{M}]\{\ddot{\mathbf{q}}_\mathrm{e}(x,t)\}+[\mathbf{K}]_0\{\mathbf{q}_\mathrm{e}(x,t)\}+[\mathbf{K}]_1\dfrac{\partial}{\partial x}\{\mathbf{q}_\mathrm{e}(x,t)\}-[\mathbf{K}]_2\dfrac{\partial^2}{\partial x^2}\{\mathbf{q}_\mathrm{e}(x,t)\}=\{\mathbf{P}_\mathrm{e}(x,t)\} \\ [\mathbf{M}]\{\ddot{\mathbf{q}}_\mathrm{s}(x,t)\}+[\mathbf{K}]_0\{\mathbf{q}_\mathrm{s}(x,t)\}+[\mathbf{K}]_1\dfrac{\partial}{\partial x}\{\mathbf{q}_\mathrm{s}(x,t)\}-[\mathbf{K}]_2\dfrac{\partial^2}{\partial x^2}\{\mathbf{q}_\mathrm{s}(x,t)\}=\{\mathbf{P}_\mathrm{s}(x,t)\} \end{cases} \tag{5}$$

2.1 Solution for displacement response of the periodic supported elastic body

To solve Eq. (5), the Fourier transform is performed, then the displacement response of the structure with moving harmonic load will be obtained.

For the periodic continuously supported single layer model, the displacement response can be expressed as,

$$\{\mathbf{q}(x,t)\}=\{\mathbf{q}_\mathrm{e}(x,t)\}+\{\mathbf{q}_\mathrm{s}(x,t)\}$$

$$=\left[\mathbf{Q}_e\left(x-x_0-ct\right)\right]\{\mathbf{P}_0\}e^{i\Omega t}-\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\mathbf{D}(\beta,\omega)\right]^{-1}\left[\mathbf{A}(\beta,\omega)\right]^{-1}\left[\mathbf{D}(\beta,\omega)\right]^{-1}e^{i\beta(x-x_0-ct)}d\beta\{\mathbf{P}_0\}e^{i\Omega t} \tag{6}$$

where,

$$\left[\mathbf{D}(\beta,\omega)\right]=-\omega^2[\mathbf{M}]+[\mathbf{K}]_0+i\beta[\mathbf{K}]_1+\beta^2[\mathbf{K}]_2 \tag{7}$$

$$\omega=\Omega-\beta c \tag{8}$$

$$\left[\mathbf{A}(\beta,\omega)\right]=\left[\mathbf{D}(\beta,\omega)\right]^{-1}+l\left[\mathbf{H}(\omega)\right] \tag{9}$$

and $\left[\mathbf{H}(\omega)\right]$ is the receptance matrix.

For the periodic discretely supported single layer model, the displacement response can be expressed as,

$$\{\mathbf{q}(x,t)\}=\{\mathbf{q}_\mathrm{e}(x,t)\}+\{\mathbf{q}_\mathrm{s}(x,t)\}$$

$$=\left[\mathbf{Q}_e\left(x-x_0-ct\right)\right]\{\mathbf{P}_0\}e^{i\Omega t}+\sum_{j=-\infty}^{\infty}\left(-\frac{e^{-i2\pi jx/l}}{2\pi l}\right)\int_{-\infty}^{\infty}\left[\mathbf{D}(\beta_j,\omega)\right]^{-1}\left[\mathbf{A}(\beta,\omega)\right]^{-1}\left[\mathbf{D}(\beta,\omega)\right]^{-1}e^{i\beta(x-x_0-ct)}d\beta\{\mathbf{P}_0\}e^{i\Omega t} \tag{10}$$

where $\left[\mathbf{D}(\beta,\omega)\right]$, ω are same with (7), (8).

$$\left[\mathbf{A}(\beta,\omega)\right]=\frac{1}{l}\sum_{j=-\infty}^{\infty}\left[\mathbf{D}(\beta_j,\omega)\right]^{-1}+\left[\mathbf{H}(\omega)\right] \tag{11}$$

2.2 Frequency dispersion equation for periodic supported elastic body

The differential equation of motion for the free vibration of the non supported elastic body is governed by,

$$[\mathbf{M}]\{\ddot{\mathbf{q}}(x,t)\}+[\mathbf{K}]_0\{\mathbf{q}(x,t)\}+[\mathbf{K}]_1\frac{\partial}{\partial x}\{\mathbf{q}(x,t)\}-[\mathbf{K}]_2\frac{\partial^2}{\partial x^2}\{\mathbf{q}(x,t)\}=0 \tag{12}$$

making Fourier transform for Eq.(12), then,

$$\left[\mathbf{D}(\beta,\omega)\right]\{\mathbf{q}(\beta,\omega)\}=0 \tag{13}$$

where $\left[\mathbf{D}(\beta,\omega)\right]$ is same with(7). In order to make sure that $\{\mathbf{q}(\beta,\omega)\}$ is not zero, there must have,

$$\det\left[\mathbf{D}(\beta,\omega)\right]=0 \tag{14}$$

This is the frequency dispersion equation of the non supported elastic body. This equation also can be generalized to the periodic supported situation, i.e.,

$$\det\left[\mathbf{A}(\beta,\omega)\right]=0 \tag{15}$$

where $\left[\mathbf{A}(\beta,\omega)\right]$ is showed in Eq.(9) and Eq.(11).

3. Model validation

Two different supported single layer beam track models with axial force are established in this paper, i.e. the continuously supported model and the periodic discretely supported model. In both models, the rail is regarded as a Timoshenko beam model for it can be used to analysis the vibration characters in high frequency range. The elastic under the rail is simplified as the equivalent spring stiffness. The temperature stress is simplified as a point force, a positive-stress means compression stress. The differential equation of motion for the Timoshenko beam subjects to an axial unit stationary harmonic load as Eq.(16).

$$\begin{cases} \rho A\dfrac{\partial^2 w}{\partial t^2}+\left(N-\kappa AG\right)\dfrac{\partial^2 w}{\partial x^2}+\kappa AG\dfrac{\partial\theta}{\partial x}=\delta\left(x-x_0-ct\right)e^{i\Omega t}+\sum\limits_{j=-\infty}^{\infty}\delta\left(x-jl\right)\left\{\mathbf{F}_j(t)\right\} \\ \rho I\dfrac{\partial^2\theta}{\partial t^2}-\kappa AG\dfrac{\partial w}{\partial x}-EI\dfrac{\partial^2\theta}{\partial x^2}+\kappa AG\theta=\sum\limits_{j=-\infty}^{\infty}\delta\left(x-jl\right)\left\{\mathbf{M}_j(t)\right\} \end{cases} \tag{16}$$

According to the wave number method for periodic structure, the matrixes in Eq.(5) can be expressed as,

$$\left[\mathbf{M}\right]=\begin{bmatrix}\rho A & 0\\ 0 & \rho I\end{bmatrix},\quad \left[\mathbf{K}\right]_0=\begin{bmatrix}0 & 0\\ 0 & \kappa AG\end{bmatrix},\quad \left[\mathbf{K}\right]_1=\begin{bmatrix}0 & \kappa AG\\ -\kappa AG & 0\end{bmatrix},\quad \left[\mathbf{K}\right]_2=\begin{bmatrix}\kappa AG-N & 0\\ 0 & EI\end{bmatrix}$$

$$\left\{\mathbf{P}\right\}=\begin{Bmatrix}\delta\left(x-x_0-ct\right)\left\{\mathbf{P}_0\right\}e^{i\Omega t}+\sum\limits_{j=-\infty}^{\infty}\delta\left(x-jl\right)\left\{\mathbf{F}_{\mathrm{j}}(t)\right\}\\ \sum\limits_{j=-\infty}^{\infty}\delta\left(x-jl\right)\left\{\mathbf{M}_{\mathrm{j}}(t)\right\}\end{Bmatrix},\quad \left\{\mathbf{q}\right\}=\begin{Bmatrix}w\\ \theta\end{Bmatrix} \tag{17}$$

In order to verify the validity of the analysis, the parametres of the rail and other components of the track structures are consistent with those in the literature [6]. In analysis, the rail type is UIC 60 and the parameters of other components under the rail are listed in Table 1. The sleeper space is 0.6m. In order to compared with [6], the vertical vibration characters of the track structure without axial force (i.e. N in the Eq.(17) is zero) are analyzed. The analyzing frequency ranges from 0 to 5000Hz.

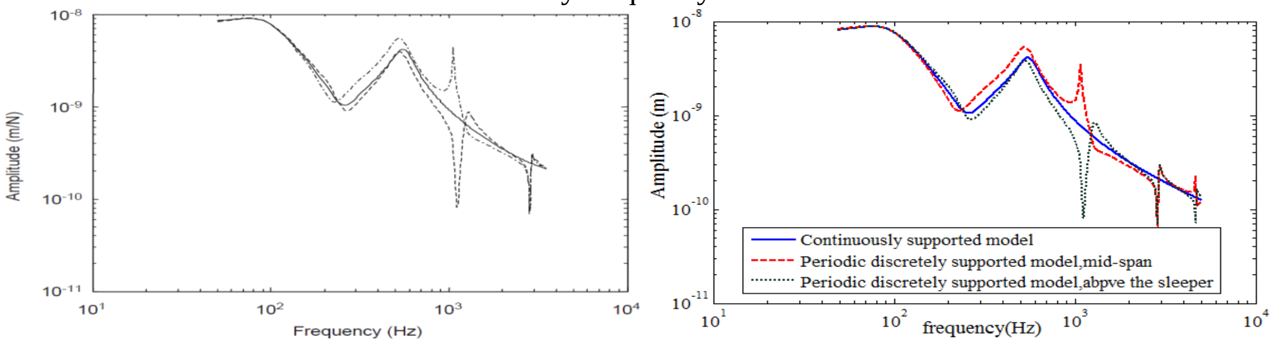
Table 1. Parameters of each component under rail

Vertical stiffness of rail pad $k_{Pv}\,(N/m)$	Loss factor of rail pad η_P	Mass of sleeper $m_S\,(kg)$	Width of sleeper $b_S\,(m)$	Vertical stiffness of ballast $k_{Bv}\,(N/m)$	Loss factor of ballast η_B
3.5×10^8	0.25	162	0.25	50×10^6	1.0

Figure 1(b) shows the displacement amplitude curves of the excitation point of continuously supported rail and periodic discretely supported rail (mid-span). Compared Figure1(b) with Figure 1(a), it can be seen that the results calculated in this paper are quite consistent with the results in literature [6].

By comparing the displacement amplitude curve of the excitation point of continuously supported rail and periodic discretely supported rail (mid-span) in Figure 1, it can be seen that the mainly difference of continuously supported model and periodic discretely supported model is the later one can be used to analyze the vibration characters above 1000Hz.

Figure 2 shows the displacement amplitude curve of the excitation point (mid-span) of the periodic discretely supported rail. According to Figure 2, it can be seen that while the analyzing frequency ranges from 0 to 5000Hz, there are about 5 resonant frequencies, i.e. A, C, D, G, H. Among them, the resonant frequency D and H are the first and second order pinned-pinned resonant frequency. While the excitation frequency near the resonant frequency of the structure, the displacement will reach a peak value that can be clearly seen in the chart. Considering this characteristic, the resonant frequencies between 0 and 5000Hz are chosen as the evaluating index in this paper, , in order to find out which is the most sensitivity frequency to the axial stress.



(a) (b)

Figure 1: Displacement amplitude curves of the excitation point of continuously supported rail and periodical discretely supported rail (mid-span): (a) results in literature [6], (b) results calculated in this paper.

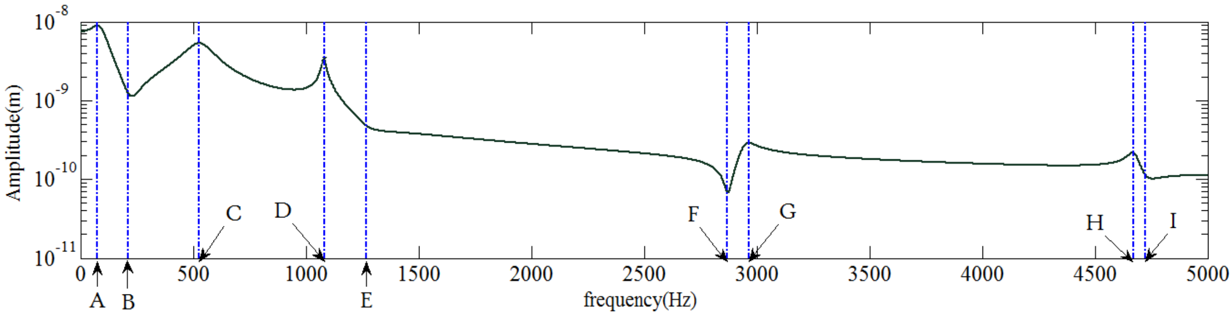


Figure 2: The displacement amplitude curve of the excitation point (mid-span) of periodic discretely supported model

4. Analysis of the effect of axial force on the resonant frequency

According to structural dynamics, the natural frequency of the beam will also change if it is subjected to an axial force. As the temperature stress is closely related to the rail temperature rise amplitude and considering the maximum temperature rise and drop amplitude which may occur in the rail, 11 different temperature rise amplitudes are selected and listed in Table 2. The rail type is UIC 60 and the parameters of other components under the rail are same with Table 1. The results are shown in Figure 4.

Table 2. Temperature increase amplitude and corresponding axial force

Temperature rise am -plitude (℃)	-50	-40	-30	-20	-10	0	10	20	30	40	50
Temperature force (MN)	-0.954	-0.763	-0.572	-0.381	-0.171	0	0.171	0.381	0.572	0.763	0.954

According to Figure 3(a)~(e), it can be seen that, all the resonant frequencies under 5000Hz are affected by the axial stress. And the resonant frequencies of the periodic discretely supported rail are increased with the increase of the axial tensile stress and decrease with the increase of the axial compressive stress. In Figure 3(f), it shows the higher the resonant frequency is, the more obvious the influence of the axial temperature force is, especially the resonant frequency D to H. The average changing rate of the resonant frequency D, G, H is 5.13Hz/MN, 7.54Hz/MN and 10.83Hz/MN, respectively.

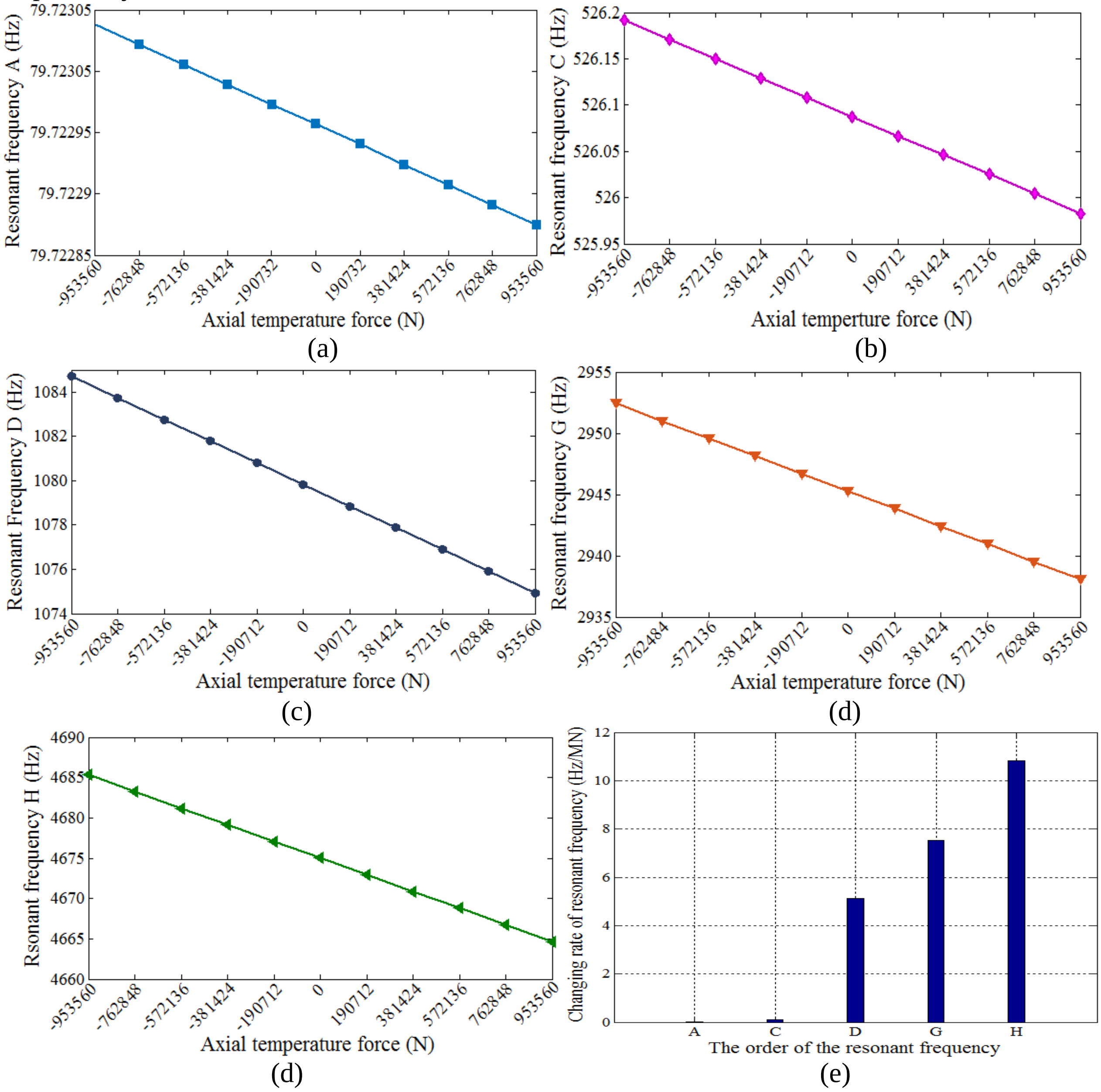


Figure 3: The effect of axial force on the resonant frequencies: (a) resonant frequency A; (b) resonant frequency C; (c) resonant frequency D; (d) resonant frequency G; (e) resonant frequency H; (f) average changing rate of the resonant frequencies.

5. Conclusions

In this paper, the vibration characters of the continuously supported rail and periodic supported rail without axial force have been discussed in order to verify the validity of the analysis process. Then the vibration characters of periodic supported rail with axial force are analyzed in order to find the relationship between them. Through the analysis above, the results obtained in this paper are consistent with the results obtained by the finite element method. The main conclusions of this paper are as follows:

Firstly, when the analyze frequency is lower than 1000Hz, the continuously supported rail model is more suitable than the periodic discretely supported rail model for the results have good consistency in the low frequency range. In addition, the operation efficiency of the continuously supported rail model is higher than that of the periodic discrete support one.

Secondly, in the analyze frequency range (0~5000Hz), all the resonant frequency of the vertical vibration of periodic supported rail are increased with the increase of the axial tensile force and decreased with the increase of axial compressive force.

Thirdly, the higher the resonance frequency is, the more obvious the influence of axial temperature force is. Especially the resonant frequency D, G,H and the average changing rate of them are 5.13Hz/MN, 7.54Hz/MN and 10.83Hz/MN.As the pinned-pinned resonant frequency is easier to be identified than the other resonant frequency, so the resonant frequency D and H can be considered as the detection indexes of the temperature stress inside the rail

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