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MODEL ORDER REDUCTION IN COMPUTATIONAL AEROACOUSTICS

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A reduced-order model for acoustic propagation is proposed. It is based on a Proper Orthogonal Decomposition (POD) of a set of snapshots, obtained for different values of geometrical and frequency parameters of the acoustic problem under study. The POD expansion coefficients, functions of the parameters, are continuously extended in the parameter space by interpolation. This approach is termed POD with Interpolation (PODI). The model is applied to the case of the scattering of sound by a circular cylinder. The problem is formulated in the frequency space and the parameter is the distance of the monopole source with respect to the circular cylinder.
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1. Introduction

Recent years have seen considerable progress in Computational AeroAcoustics (CAA). Models of acoustic propagation based on the Linearized Euler equations (LEE) and Linearized Navier-Stokes equations (LNSE) are now currently in use [1, 2, 3], leading to advances across a broad range of engineering applications. Improvements in methodology, together with a substantial increase in computing power, are such that real-time simulations and optimization of systems governed by LEEs, or LNSEs, is now an attainable goal. In many cases, computational models for such applications yield very large systems that are computationally intensive to solve. A critical element towards achieving a real-time simulation capability is the development of accurate and efficient models that can be solved sufficiently rapidly.

Model reduction is a powerful tool that allows the systematic generation of cost-efficient representations of large-scale systems resulting from discretization of high-fidelity models. Reduction methodologies have been developed and applied for many different disciplines, including controls, fluid dynamics, structural dynamics, and circuit design. Considerable advances in the field of model reduction for large-scale systems have been made and many different applications have been demonstrated with success [4].

The choice of the particular Reduced-Order Model (ROM) however is quite critical, as it must preserve the essential physics and predictive capability of the high-fidelity partial differential model. The ROM definition can be sample based, employing statistical analysis, such as kriging, or based on dimensionality considerations: if the solution of the partial differential problem evolves in a low-dimensional manifold induced by the parametric dependence, the high dimensionality of the discretization space can be reduced constructing an approximation of this manifold. Among the methods based on dimensionality reduction, the Proper Orthogonal Decomposition (POD) is often used for reduced order approximations. Given a series of high-fidelity snapshots of a system, a simple algorithm produces an optimal linear basis which may be interpreted as a solution of a minimization of the projection error of the original system, as described by the snapshots, equivalent to maximizing the energy in the projection [5]. If a problem is described by a representative number of high-fidelity calculations from which a set of basis vectors may be extracted, the singular values become rapidly small and a small number of basis vectors is sufficient to approximate the solution in terms of a set of low rank basis vectors. The ROM is obtained using this reduced set. This linear basis can then be used to compute new approximate solutions for arbitrary parameters at a significantly reduced computational cost.

In fluid dynamics, the POD is usually employed to find a basis for the projection of the Navier-Stokes equations and to obtain a ROM composed by a system of ordinary differential equations for the time dependent POD expansion coefficients [6]. Less commonly, the POD is applied in the frequency space [7] or in a parameter space. In this latter case, as an example, the POD can be used to describe flow fields around modified body shapes, using the information about the flow past few selected geometries of the body, which form the snapshots for the POD. Examples of this approach can be found in the works of Legresley and Alonso [8], Bui-Thanh et al. [9], Mifsud et al. [10] and Tang and Shyy [11]. The POD expansion coefficients are functions of the parameters, and can be continuously extended in the parameter space by the Response Surface Method (RSM) [12]. This approach is termed POD with Interpolation (PODI) [9].

The application of POD-based ROMs in the parameter space is quite recent and still under active development. Moreover, to our knowledge, no systematic study of model reduction has been presented in aeroacoustics yet, addressing a number of important issues, including the reliability of reduction techniques, associated with the quality of the reduced models, and validity of the model over a range of operating conditions.

In the next section the PODI method is described, and in Section 3 it is applied to the problem of the scattering of sound by a circular cylinder. The problem is formulated in the frequency space and the parameter is the distance of the monopole source with respect to the circular cylinder.

2. Proper Orthogonal Decomposition

Several POD methods can be found in literature: the Karhunen-Loève decomposition, the Principal Component analysis and the Singular Value Decomposition (SVD). It can be shown that they are all equivalent [5].

POD basis vectors are computed using a set of m high-fidelity data called snapshots. m is the number of realizations obtained combining different values of the parameters spanning the parameter space θ, such as parameters defining the domain geometry or the position of the noise sources. Each snapshot corresponds to a vector of dimension n, solution of a high-fidelity CAA calculation, where $n \gg m$. This input choice is critical, since the resulting basis will capture only those dynamics present in the snapshot ensemble.

In this work the POD snapshots are obtained from a frequency formulation of the CAA model. The basis vectors $\boldsymbol{\varphi}$ are chosen so as to maximize the functional

$$\boldsymbol{\varphi} = \arg\max \frac{|(\mathbf{u}, \boldsymbol{\varphi})|^2}{(\boldsymbol{\varphi}, \boldsymbol{\varphi})} \ , \tag{1}$$

where $(\mathbf{u}, \boldsymbol{\varphi})$ denotes the scalar product of the basis vector with the field $\mathbf{u}(\theta, k)$, which depends on parameter space θ and wave number k. A necessary condition for Eq.(1) to hold is that $\boldsymbol{\varphi}$ is an eigenfunction of the kernel K defined by

$$K(\theta, \theta') = \mathbf{u}(\theta, k)\mathbf{u}^*(\theta', k) \ ,$$

where $\mathbf{u}^*$ denotes the complex conjugate transpose of $\mathbf{u}$. Instead of explicitly calculating the kernel

K it is possible to adopt the snapshot method [13], approximating the kernel as

$$K(\theta, \theta') = \frac{1}{m} \sum_{i=1}^{m} \mathbf{u}_i(\theta) \mathbf{u}_i^*(\theta') \ ,$$

where $\mathbf{u}_i(\theta) = \mathbf{u}(\theta_i, k)$ is the snapshot corresponding to the i-th parameter configuration and the number of snapshots m is sufficiently large. The basis $\boldsymbol{\varphi}$, eigenvectors of K, are of the form

$$\boldsymbol{\varphi} = \sum_{i=1}^{m} \beta_i \mathbf{u}_i \ ,$$

where the constants β_i satisfy the eigenvector equation

$$R\boldsymbol{\beta} = \Lambda \boldsymbol{\beta} \ ;$$

R is the correlation matrix

$$R_{ij} = \frac{1}{m} (\mathbf{u}_i, \mathbf{u}_j) \ ,$$

and Λ is the diagonal matrix with eigenvalues λ_j. The magnitude of the j-th eigenvalue describes the relative importance of the $j-$th POD basis vector for reconstruction of the data contained in the snapshot ensemble. For a basis containing the first p POD modes, an heuristic criterion based on the ratio of the modeled to the total energy contained in the system is

$$\varepsilon(l) = \frac{\sum_{i=p}^{l} \lambda_i}{\sum_{i=1}^{m} \lambda_i} \ .$$

The vector $\mathbf{u}_j \in \mathbb{R}^n$ represents a vector of scalar functions of grid points (or cells), such as the acoustic pressure. Each snapshot $\mathbf{u}_j$ can be expanded as

$$\mathbf{u}_j = \sum_{l=1}^{m} \alpha_{lj} \boldsymbol{\varphi}_l \quad \text{for } j = 1, \ldots, m \ ,$$

where the projection coefficients

$$\alpha_{lj} = (\boldsymbol{\varphi}_l, \mathbf{u}_j) \ ,$$

are discrete functions in the parameter space, with values defined at the points corresponding to the individual snapshots $\mathbf{u}_j$. To use the derived ROM as a prediction tool, it is necessary to extend the discrete functions α_{lj} in continuous functions α_l in the parameter space, and the field variable in a generic point of the parameter space may be approximated by the linear combination

$$\mathbf{u} = \sum_{l=1}^{m} \alpha_l \, \boldsymbol{\varphi}_l \ .$$

3. Numerical example: acoustic scattering

The PODI-ROM method described in the previous section is applied to a problem of acoustic propagation in a homogeneous medium at rest, formulated in the frequency domain. The wave equation, in Cartesian coordinates and assuming the usual index summation convention,

$$\frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} - \delta_{ij} \frac{\partial^2 p}{\partial x_i \partial x_j} = -Q_{ac} \quad , \tag{2}$$

describes the propagation of sound through a homogeneous medium at rest. p is the acoustic pressure, c_0 the sound speed evaluated at the medium conditions p_0 and ρ_0, constant through the medium. $Q_{ac}(t, x_i)$ represents a source term (for example a line monopole source). Assuming harmonic time dependence for the acoustic fluctuations

$$p = \widehat{p}\; e^{I\omega t} \quad ,$$

with $I = \sqrt{-1}$ and ω being the angular frequency, Eq.(2) transforms in the complex Helmholtz equation

$$\delta_{ij} \frac{\partial^2 \widehat{p}}{\partial x_i \partial x_j} + k^2 \widehat{p} = \widehat{Q}_{ac} \quad , \tag{3}$$

with wave number $k = \omega/c_0$. The acoustic pressure can be computed after the inhomogeneous term Q_{ac} of Eq.(2) has been evaluated and Fourier transformed $(\widehat{Q}_{ac}(k, x_i))$. In this way the range of the wave number $k \in [0, k_{max}]$ is introduced. For each value of k is associated the Helmholtz problem (3) with the corresponding inhomogeneous forcing term $\widehat{Q}_{ac}(k, x_i)$. The time dependent acoustic pressure field is recovered performing an inverse DFT. Problem(3) is solved with a finite element method applying the code FreeFem++ [14].

The scattering of sound by a circular cylinder is a useful test for the validation of the PODI-ROM technique. The exact solution of the scatter of sound from a monopole line source, positioned at a distance L, along the x-axis ($x_{source} = L, y_{source} = 0$), from a circular cylinder, centered in the origin of the axis, with ray R, has been provided by Morris[15]. The corresponding Helmholtz problem (3) has been solved on a square domain $[-2\pi : 2\pi] \times [-2\pi : 2\pi]$. To avoid spurious reflections along the exterior boundaries, Perfectly Matched Layer (PML) boundary conditions are imposed [16]. PML regions are added outside the exterior boundaries, and fully reflecting conditions ($\partial p/\partial n = 0$) are imposed along the cylinder wall.

Keeping a fixed wave number $k = 10$, corresponding to a frequency of 541.397 Hz, several snapshots are obtained varying the distance of the line source with respect to the cylinder center. With an unstructured grid of 36310 triangles, the numerical solution coincides with the analytical solution within round-off accuracy outside the source region.

Five snapshots are obtained varying the distance L/R: 1.50, 2.0, 2.5, 3.0 and 3.50. Once the POD coefficients are obtained, the PODI-ROM is applied to reconstruct the solution for the case $k = 10$ and $L/R = 2.75$. The POD coefficient are linearly interpolated to obtain the values corresponding to the chosen parameter L/R. In Fig. 1 the comparison between the reconstructed solution and the numerical solution is reported. The agreement is quite good. The pressure field is slightly overestimated in the region $1 < x/R < 2.75$. Moreover in the source region some discrepancies are evident. The main reason could be the presence of the source term in the numerical model. The reconstructed acoustic field is very similar to the numerical one, as shown in Fig. 4, where the instantaneous pressure field is reported, and in Fig. 4 displaying the SPL.

4. Conclusions

A reduced-order model for acoustic propagation has been proposed. It is based on a POD analysis of a set of snapshots, obtained for different values of geometric and frequency parameters of the acoustic problem under study. The POD expansion coefficients, functions of the parameters, are continuously extended in the parameter space by interpolation. This approach is termed POD with Interpolation (PODI). The model is applied to the case of the scattering of sound by a circular cylinder. The problem is formulated in the frequency space and the parameter is the distance of the monopole source with respect to the circular cylinder. This preliminary test shows the potentiality of the method. Further analysis are needed to asses the accuracy of the method.

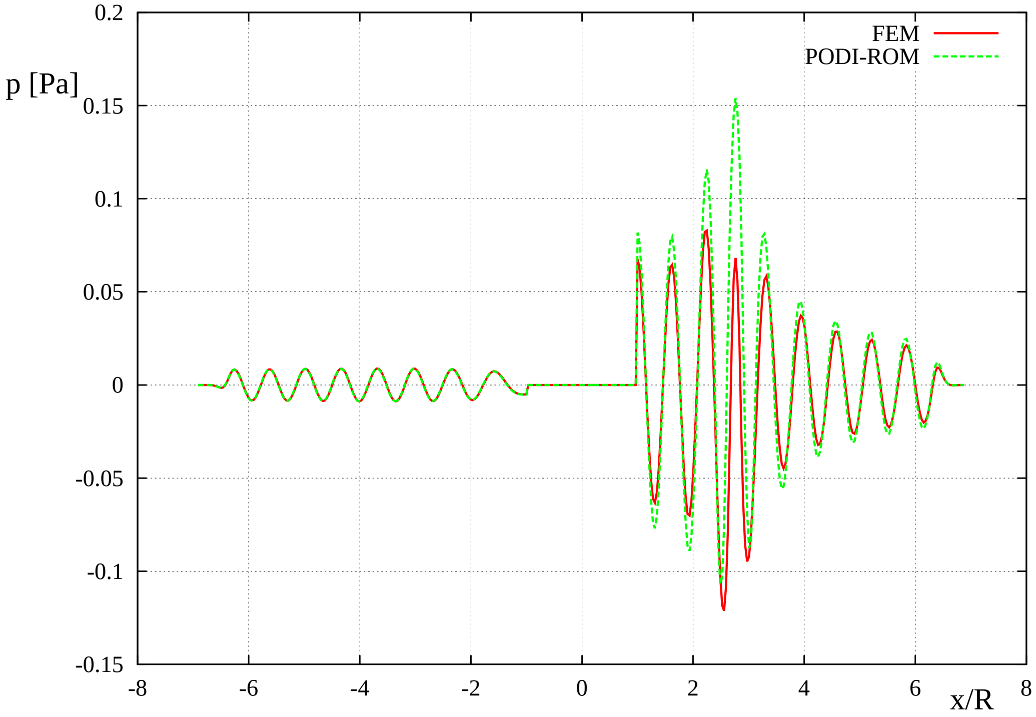
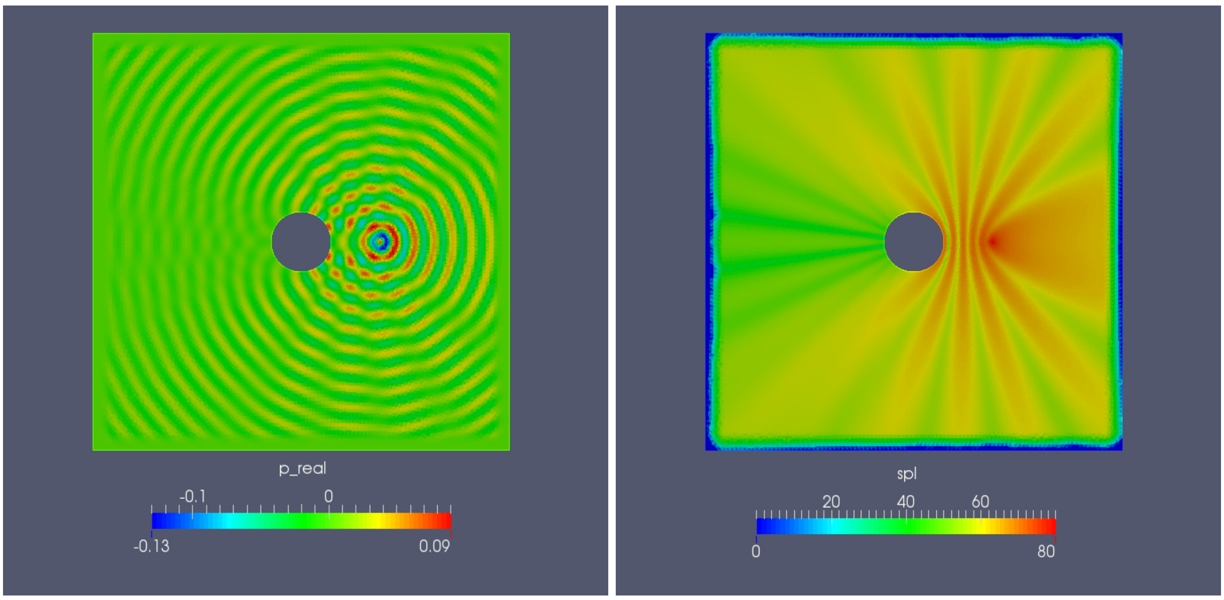


Figure 1: Scattering of a monopole acoustic source from a circular cylinder. Instantaneous pressure, along the line $y = 0$, $k = 10$. Comparison of the PODI-ROM reconstruction with the numerical (FEM) solution.



(a) (b)

Figure 2: Scattering of a monopole acoustic source from a circular cylinder. $k = 10$. (a) Instantaneous pressure [Pa] and (b) SPL [dB] of the PODI-ROM solution.

REFERENCES

1. Kierkegaard, A., Boij, S. and Efraimsson, G., A frequency domain linearized Navier-Stokes Equations Approach to Acoustic Propagation in Flow Ducts with Sharp Edges, *The Journal of the Acoustical Society of America*, **127** (2), 710–719, (2010).

2. Arina, R., Validation of a Discontinuous Galerkin Implementation of the Time-Domain Linearized Navier-Stokes Equations for Aeroacoustics, *Aerospace*, **3**, (7) doi:10.3390/aerospace3010007, (2016).

3. Lario, A and Arina, R., Linearized Navier-Stokes Equations and their Numerical Solution, *22nd AIAA/CEAS Aeroacoustics Conference; Lyon, France*, AIAA Paper 2016-2973, (2016).

4. Chinesta, F., Huerta, A., Rozza, G. and Willcox, K., Model Order Reduction, *Encyclopedia of Computational Mechanics*, Wiley, (2016).

5. Liang, Y. C., Lee, H. P., Lim, S. P., Lin, W. Z. and Lee, K. H., Proper Orthogonal Decomposition and its Applications - Part I: Theory, *J. Sound and Vibration*, **252** (3), 527–544, (2002).

6. Rowley, C. W., Colonius, T. and Murray, R. M., Model Reduction fo Compressible Flows Using POD and Galerkin Projection, *Physica D*, **189**, 115–129, (2004).

7. Epureanu, B. I., A Parametric Analysis of Reduced Order Models of Viscous Flows in Turbomachinery, *J. of Fluids and Structures*, **17**, 971–982, (2003).

8. Legresley. P. A. and Alonso, J. J., Airfoil Design Optimization Using Reduced Order Models Based on Proper Orthogonal Decomposition, AIAA 2000-2545, *Fluids 2000 Conference*, Denver, Colorado, June (2000).

9. Bui-Thanh, T., Damodaran, M. and Willcox, K., Aerodynamic Data Reconstruction and Inverse Design Using Proper Orthogonal Decomposition, *AIAA Journal*, **42** (8), 1505–1516, (2004).

10. Mifsud, M. J., Shaw, S. T. and MacManus, D. G., A High-fidelity Low-cost Aerodynamic Model Using Proper Orthogonal Decomposition, *Int. J. Numer. Meth. Fluids*, **63**, 468–494, (2010).

11. Tang, L. and Shyy, W., Proper Orthogonal Decomposition and Response Surface Method for TPS/RLV Structural Design and Optimization: X-34 Case Study, AIAA 2005-0839, *43rd AIAA Aerospace Sciences Meeting*, Reno, Nevada, January (2005).

12. Myers, R. H., Montgomery, D. C. and Anderson-Cook, C. M., *Response Surface Methodology*, 3rd ed., Wiley, New York, (2009).

13. Sirovich, L., Turbulence and the Dynamics of Coherent Structures, Part 1: Coherent Structures, *Quarterly of Applied Mathematics*, **45** (3), 561–590, (1987).

14. Hecht, F., New Developments in FreeFem++, *J. Numer. Math.*, **20** (3-4), 251–265, (2012).

15. Morris, P.J., The Scattering of Sound from a Spatially Distributed Axisymmetric Cylindrical Source by a Circular Cylinder, *J. Acoust. Soc. Am.*, **97** (5), 2651–2656 (1995).

16. Harari, I., Slavutin, M. and Turkel, E., Analytical and Numerical Studies of a Finite Element PML for the Helmholtz Equation, *J. Comput. Acoustics*, **8** (1), 121-137 ,(2000).