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NEW BEAM ELEMENTS

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Introduction. The stiffness and mass matrices for the simple beam finite element were published in 1963 by Leckie and Lindberg[1]. Their analysis was restricted to beams which had large length to depth ratios. More recently Archer[2] published a stiffness and consistent mass matrix for a Timoshenko beam, i.e. one in which shear deformation and rotary inertia are included. Kapur[3] derived stiffness and mass matrices for a Timoshenko beam in which four degrees of freedom at each node were required for considering unidirectional bending of a straight beam.

The in-plane and out of plane stiffness matrices for a 'thin' curved beam were derived by Martin[4] and Lee[5] respectively. The curved Timoshenko beam (with certain restrictions) has been studied by Seidal and Erdelyi[6] and Rao and Sundararajan[7] using other methods.

This paper describes the derivation of stiffness and consistent mass matrices for straight and curved Timoshenko beams and a curved beam which does not include shear deformation and rotary inertia. In all these cases, the element analysis is based upon the exact differential equations for static motion and includes the effects of bending in both principal directions, extension and torsion.

A further extension, the 'off-set' beam , allows complicated cross section beams in which shear and flexural centres are not coincident to be built up. This element can be added to a plate structure in such a way that the effects of stiffener eccentricity can be allowed for.

All of the elements derived are part of the PAFEC 70[8] suite of Fortran subroutines which enables general three dimensional structures to be analysed.

Theoretical Method. All the elements are based on the exact differential equations for static motion and the method of solution is the same. The necessary steps in the analysis are as follows:

(a) Static equilibrium relations and equations connecting stresses and displacements for an infinitesimal element are obtained.

(b) The complete general solutions to the differential equations in (a) are found in terms of constants of integration, beam properties and coordinates.

(c) The generalised forces and displacements acting at the nodes of the finite element are then substituted into the solutions. The element stiffness matrix is then obtained by eliminating the constants of integration from the two resulting equations.

(d) The kinetic energy of the infinitesimal element is written down and a sinusoidal variation of displacements with time is assumed. The solutions for the displacement of the beam along its length are then substituted in the kinetic energy expression which is integrated along the length of the beam. This kinetic energy is then differentiated with respect to the displacements to give the element mass matrix.

To formulate the stiffness and mass matrices for the off-set beam a simple transformation is applied to the matrices of the simple beam element. The transformation matrix, which allows the effects of forces and displacements acting at distances away from the element nodes to be examined, is easily obtained using equilibrium and continuity conditions.

Results. A selection of results is given for the various elements.

(1) Vibration of a 'thick' single bay portal frame (see fig. 1) using the straight Timoshenko beam.

Frequencies for the fundamental mode (symmetric out of plane) of the portal frame are given in Table 1. The exact results were obtained using the dynamic stiffness matrix method[10], both with and without shear deformation and rotary inertia.

Table 1. Fundamental frequency (Hz) for 'thick' single bay portal frame.

Elements/leg (degrees of freedom/leg)				Exact Timoshenko	Exact Bernoulli-Euler
1(12)	2(30)	3(48)	4(66)		
531.024	528.789	528,481	528,383	528.264	551.703

(2) In-plane vibration of a circular ring using curved beams with and without shear deformation and rotary inertia. The ring analysed had the following properties. Radius of ring = 10in. Radius of cross-section of ring = .1in. $E = 3 \times 10^7 \text{lbf/in}^2$. $\rho = 283 \text{ lbm/in}^3$

Using symmetry the analysis was carried out on a quarter circle. Frequencies are given for two circumferential waves in Table 2.

Table 2. Fundamental frequency (Hz) of in-plane vibrations of a thin ring

	No. of elements (degrees of freedom)			Timoshenko[11] inextensional
	2(5)	4(11)	6(17)	
a	43.2117	43.2088	43.2079	
b	43.2189	43.2162	43.2151	43.2158

Row a - curved beam with shear deformation and rotary inertia
Row b - curved beam without shear deformation and rotary inertia

(3) Out of plane vibration of a clamped/clamped circular arc using a curved beam element without shear deformation and rotary inertia. The properties of the ring are similar to those in example 2. No account was taken of the symmetry of the arc and frequencies are given in Table 3 for the fundamental mode for an arc subtending an angle of 180°. The frequencies are compared with the approximate results due to Brown[12].

Table 3. Fundamental frequencies (Hz) of out of plane vibrations of a circular arc

No. of elements (degrees of freedom)			Approx. frequency
2(3)	4(9)	6(15)	
29.691	29.374	29.353	29.257

(4) Eccentrically stiffened plates using the off-set beam. Frequencies were computed for a simply supported rectangular plate (see Fig. 2) with one, two and three stiffeners. The results obtained were compared with values due to Long[13] which took the eccentricity of the stiffener into account.

When the off-set beam element was being used the stiffener was off-set by the ammount (e) from the middle surface of the plate. With an eccentrically stiffened plate the neutral axis of the combined system moves in the direction of the stiffener and stretching of the plate/stiffener must be taken into account. This was achieved by superimposing eight node bending[14] and membrane[15] elements. The frequencies were computed using both the off-set beam and the simple beam to show the effects of neglecting off-set. The results for the fundamental frequency are shown in Figs. 3, 4 and 5.

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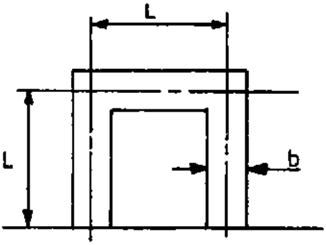

Fig. 1 Portal Frame

$L = 10in.$

$b = t = 3in.$

$\rho = .283\ lbm/in^3.$

$\frac{E}{GK} = 3.125$

(where K is a shear constant[9])

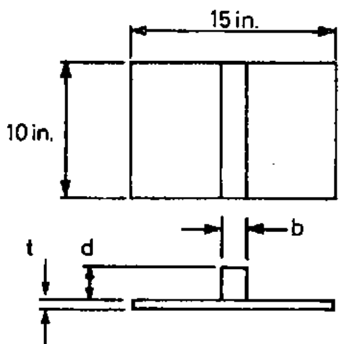

Fig. 2 Stiffened Plate

$E = 3 \times 10^7 lbf/in^2.$

$\nu = .3$

$\rho = .283\ lbf/in^3.$

$t = .1in.$

$b = .25in.$

$e = (d + t)/2$

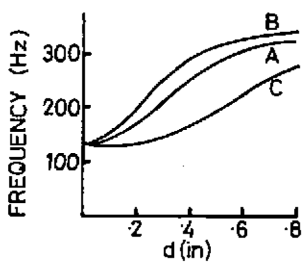

Fig. 3 One Stiffener

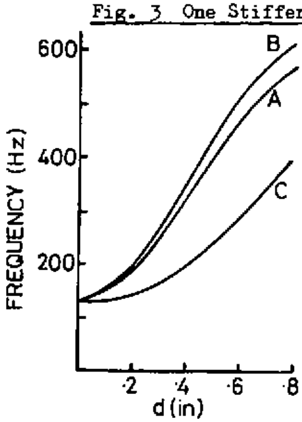

Fig. 4 Two Stiffeners (Equally spaced)

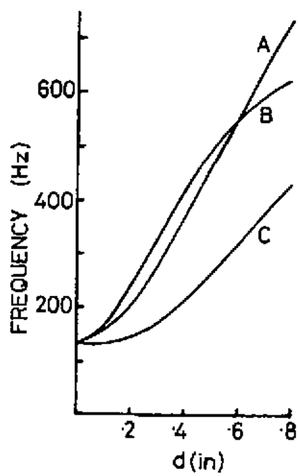

Fig. 5 Three Stiffeners (Equally spaced)

Curve A Long's results.
Curve B With off-set beam.
Curve C With simple beam.