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SOUND PROPAGATION IN RANDOM MEDIA

A new theory generalizing the Parabolic Equation Method

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1. INTRODUCTION

Sound waves have already been used for probing meteorological data such as large scale temperature and velocity profiles /1/. To extend this kind of remote sensing technique to small scale temperature and velocity inhomogeneities generated by the turbulent motion of the air a statistical theory is needed. Although no general theory to connect sound pressure statistics and atmospheric statistics exists, there are several approximate solutions to the problem. The most common approximation is the *parabolic equation method*. Results obtained by this method are restricted to small fluctuations of a scalar refractive index and to small wave lengths. While the atmospheric fluctuations are almost small, it is desirable to overcome the wave length restriction. Furthermore, the temperature and wind field cannot be reduced to a scalar refractive index in general. This paper presents a twofold generalization of the parabolic equation method. At first, the small wave length restriction is removed by the developement of an approximation method based on the scalar Helmholtz equation instead of its parabolic form. At second, the sound wave equation given by Tatarskii /2/, which contains the whole velocity vector, is transformed into an operator Helmholtz equation on which the approximation method can be applied, too. In this paper we confine ourselves to the first statistical moment of the wave field. The proposed method can also be applied to higher statistical moments. References will be given for more general results.

2. TEMPERATURE FLUCTUATIONS

We start with the scalar Helmholtz equation containing a random function μ (refractive index deviation from its mean value)[1]:

$$\left(\Delta + k^2(1 + \mu(\vec{r}))\right)\ \psi(\vec{r}) = 0 \tag{1}$$

Δ = *Laplace operator;* k = *wave number;* ψ = *complex sound pressure*
Equation (1) is a stochastic differential equation and the sound pressure ψ becomes a random function, too. Stochastically this equation is nonlinear, e.g. it contains a product of two random

[1] It can be seen from the more general wave equation (15) that μ is a function of temperature, if the velocity vector is neglected.

variables. For this reason the stochastic Helmholtz equation cannot be solved exactly. Further approximations are necessary. Mainly two kinds of smallness are used for the various approximation methods. Temperature fluctuations are limited by the condition $<\mu^2> \ll 1$ and the wave length is assumed to be much smaller than the typical size of the mediums inhomogeneities expressed by the correlation length ($\lambda \ll l$). While the parabolic equation method is based on both assumptions, the generalized method described in section 2.2 replaces the small wave length limit by the weaker condition of neglecting the backscattering.

2.1 Parabolic Equation Method

The widely used parabolic equation method solves the stochastic Helmholtz equation for the statistical moments of the wave function. It is briefly reviewed here in order to compare its results with corresponding results of our generalization. For the small wave length limit the Helmholtz equation can be converted approximately into a parabolic form using $\psi = \varphi e^{ikz}$ /4/:

$$\left(2ik\frac{\partial}{\partial z} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\mu(\vec{r})\right) \quad \varphi(\vec{r}) = 0 \tag{2}$$

This approximation physically means the restriction to small scattering angles. The wave propagates within a narrow cone in the main propagation direction z. The smallness of $<\mu^2>$ is used to derive moment equations by the *local method of small perturbations* /4/. By this method the scattering volume is divided into slabs perpendicular to the main propagation direction z. Each slab is chosen as thin as required by the validity limit of the first order perturbation expansion term (single scattering approximation, Born approximation). This distance clearly depends on the strength of the temperature fluctuations. If these fluctuations are sufficiently small, the slabs are much thicker than one correlation length of the random medium. Therefore the slabs can be regarded as uncorrelated. Based on both assumptions - small temperature fluctuations and a small wave length compared to the correlation length - the statistical independence of subsequent slabs can be proofed mathematically. Wave propagation through random media is described here as a Markov process. As a consequence of the Markov property slabs of finite thickness are no longer necessary. This results in a differential equation for the mean wave function[2]:

$$\left(2ik\frac{\partial}{\partial z} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{ik^3}{4}A_\mu(0,0)\right) \quad <\varphi(\vec{r})> = 0 \tag{3}$$

$$A_\mu(x,y) := \int_{-\infty}^{\infty} dz\, B_\mu(x,y,z) \quad , \quad B_\mu(\vec{r}) := <\mu(\vec{r}_0)\mu(\vec{r}_0+\vec{r})> \tag{4}$$

Equation (2) can be solved easily. Using again $\psi = \varphi e^{ikz}$, it follows:

$$<\psi(\vec{r})> = \psi_0(\vec{r})\, exp\left\{-\frac{k^2}{8}A_\mu(0,0)\, z\right\} \tag{5}$$

[2] The autocorrelation function B_μ is assumed to be homogeneous and isotropic. Equations for the higher statistical moments obtained by this method are found in /2/ and /4/.

The mean complex sound pressure decreases exponentially while the wave propagates through the random medium. This is an effect of decorrelation of the sound wave due to phase fluctuations. Different members of the statistical assembly interfere destructively because of their different phases. The validity of result (5) - and of any other result obtained by the parabolic equation method - is restricted first by the validity of the parabolic wave equation (2) and second by the validity of the Markov assumption. Necessary conditions are the smallness of temperature fluctuations and the smallness of the wave length (compared with the correlation length). In the next section the first condition is also assumed to be true. The small wave length assumption, however, will be replaced by the weaker condition of negligible backscattering. This will lead to a generalized Markov process and, consequently, to generalized results with respect to the wave length - correlation length ratio.

2.2 Generalized Local Method of Small Perturbations

While generalizing the parabolic equation method the main idea of the local method of small perturbations will be retained. The temperature fluctuations are assumed to be small enough to justify the application of the Born approximation within a distance Δz in the scattering volume which is large compared to the correlation length. Again the scattering volume is divided into slabs of this size. Therefore subsequent slabs are uncorrelated as well. Neclecting the backscattering yields a difference equation for the mean sound pressure:

$$<\psi(\vec{\rho}_n, n\Delta z)> = \left(\hat{G}_0 + \hat{S}\right) <\psi(\vec{\rho}_{n-1}, (n-1)\Delta z)> \qquad (6)$$

$\vec{\rho} = (x, y)$; n = *number of slab*

$\hat{G}_0$ is the integral operator of the homogeneous Helmholtz equation ($\mu = 0$, free propagation) and $\hat{S}$ is an integral operator for the scattering within one slab. Since double scattering is the lowest order non-vanishing term, the kernel of $\hat{S}$ contains the autocorrelation function of the temperature field:

$$\hat{S}(\vec{r}_n, \vec{r}) <\psi(\vec{r})> = -k^4 \int d^3\vec{r}' \int d^3\vec{r}\, G(\vec{r}_n, \vec{r}')G(\vec{r}', \vec{r}) <\mu(\vec{r}')\mu(\vec{r})><\psi(\vec{r})> \qquad (7)$$

G = *Greens function of the Helmholtz equation*

For the case of a homogeneous temperature autocorrelation function, equation (7) becomes a convolution product. It can be Fourier-transformed with respect to the variable $\vec{\rho}$, and this operation turns the operators $\hat{S}$ and $\hat{G}_0$ into simple functions $\tilde{S}$ and $\tilde{G}_0$. In the Fourier representation equation (6) reads:

$$<\tilde{\psi}(\vec{\kappa}, n\Delta z)> = \left(\tilde{G}_0 + \tilde{S}\right) <\tilde{\psi}(\vec{\kappa}, (n-1)\Delta z)> \qquad (8)$$

$\vec{\kappa}$ = *2-dimensional spatial frequency*

By Fourier-transformation the $\vec{\rho}$-integrations are already performed. After z-integration the scattering contribution of one slab is seen to be approximately proportional to the slabs thickness Δz. Therefore equation (8) can be written as ($\widetilde{S} = \widetilde{s}\ \Delta z$):

$$<\widetilde{\psi}\,(\vec{\kappa}, n\Delta z)> = \left(\widetilde{G}_0 + \widetilde{s}\ \Delta z\right)\ <\widetilde{\psi}\,(\vec{\kappa},(n-1)\Delta z)> \qquad (9)$$

The effect of all slabs is obtained by iteration:

$$<\widetilde{\psi}\,(\vec{\kappa}, n\Delta z)> = \left(\widetilde{G}_0 + \widetilde{s}\ \Delta z\right)^n\ \widetilde{\psi}_0\,(\vec{\kappa},0) \qquad (10)$$

Because of the smallness of the scattering contribution to one slab the product in (10) can be expressed by an exponential function ($z = n\,\Delta z$):

$$<\widetilde{\psi}\,(\vec{\kappa}, z)> = \widetilde{\psi}_0\,(\vec{\kappa}, z)\ exp\left\{\widetilde{s}\,(\vec{\kappa})\,z\right\} \qquad (11)$$

The scattering function $\widetilde{s}$ depends on the *3-dimensional* Fourier transform of the autocorrelation function (power spectrum) Φ_μ /5/:

$$\widetilde{s}\,(\kappa) = -\frac{k^4}{8a}\int d^2\vec{\kappa}'\ \frac{\Phi_\mu(\vec{\kappa}', a'-a)}{a'} \qquad (12)$$

$$a = \sqrt{k^2-\kappa^2}\quad,\quad a' = \sqrt{k^2-(\vec{\kappa}-\vec{\kappa}')^2}\quad,\quad \Phi_\mu(\vec{K}) := \int d^3\vec{r}\,e^{i\vec{K}\vec{r}}\,B_\mu(\vec{r}) \qquad (13)$$

To compare (11) with the corresponding parabolic equation method result (5), $\widetilde{s}$ has been calculated for an exponential autocorrelation function of the medium, which leads to a closed form expression /5/. If the incident wave ψ_0 is assumed to be a plane one, the inverse Fourier transform finally results in:

$$<\psi(\vec{r})> = \psi_0(\vec{r})\ exp\left\{-\frac{<\mu^2> k^2 l}{4}\,\frac{(k^2l^2 - ikl)}{(1+k^2l^2)}\ z\right\} \qquad (14)$$

l = correlation length

The real part of the exponent describes the decorrelation of the sound wave. It is a more general expression than $k^2 A_\mu(0,0)/8$ - only in the small wave length limit they are equal[3]. By the imaginary part of the exponent, a second effect is predicted by this method, which cannot be seen in the parabolic results. The imaginary part is a stochastic correction to the wave number k due to an increase of the mean propagation distance in the random medium. Only in the small wave length limit the scattering angles are small and the mean propagation distance correponds to the z-extension of the scattering volume. The approximation method presented here has been applied to the scalar Helmhotz equation. It will be used to solve a more general wave equation, which contains temperature and wind vector in the following section.

[3] For an exponential autocorrelation function we have $A_\mu(0,0) = 2 <\mu^2> l$.

3. TEMPERATURE AND WIND FLUCTUATIONS

Up to this point, we have discussed the wave propagation in a random medium, in which the refractive index is a scalar variable (temperature). In this section, our attention will be turned to Tatarskii's equation which includes the wind vector. At first, Tatarskii's equation is rewritten in a form that is similar to the scalar Helmholtz equation. (We call it Operator Helmholtz Equation (OHE), because the scalar refractive index is replaced by a differential operator.) At second, this equation will be reduced and solved by the parabolic approximation. It will be seen that in this case only the wind component along the direction of the incident wave affects the sound field. Finally, the equation for the first statistical moment based on OHE is derived and solved by using the generalized LMSP.

3.1 Derivation of Operator Helmholtz Equation

Starting with Tatarskii's equation /2/:

$$(\Delta + k^2)\,\psi = -\frac{\partial}{\partial x_i}\left(T'\frac{\partial\psi}{\partial x_i}\right) + \frac{2i}{k}\frac{\partial^2}{\partial x_l \partial x_i}\left(u'_l\frac{\partial\psi}{\partial x_i}\right) \tag{15}$$

where $T' = \frac{T-<T>}{<T>}$, relative temperature flucturations, $u'_l = \frac{u_l}{c_0}$: the l'th component of the relative wind vector (related to the sound speed c_0). Since the constant air motion leads merely to a Doppler frequency shift, we restrict our analysis to the case where the mean value of $\vec{u}$ is zero, $< u'_l >= 0$. (In this paper the double same subscripts represent summation.) Neglecting the higher order term of the small variable T', equation (15) can be rewritten in the following form:

$$\Delta\psi + k^2\psi = k^2 \mathbf{L}\psi \tag{16}$$

$$\mathbf{L} = T' - \frac{1}{k^2}\frac{\partial \mathbf{L}_0}{\partial x_i}\frac{\partial}{\partial x_i}, \quad \mathbf{L}_0 = T' - \frac{2i}{k}u'_l\frac{\partial}{\partial x_l} \tag{17}$$

In comparison of (16) with (1) the refractive index μ is replaced by a complex operator $\mathbf{L}$. Therefore, equation (16) is called Operator Helmholtz Equation.

3.2 Parabolic Approximation to the OHE

When the incident wave length is small in comparison with the size of the typical inhomogeneities of the medium, the wave propagates mainly along the incident wave direction. In this case, Tatarskii's equation can be recasted into a parabolic form /6/:

$$\left(2ik\frac{\partial}{\partial z} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2k^2 n\right)\varphi(\vec{r}) = 0 \tag{18}$$

$$n = -\left(1 - \frac{i}{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{2}T' + u'_z\right) \tag{19}$$

This equation is different from (2) only in the refractive index n. Here the refractive index includes not only the temperature, but also the wind velocity. Due to the small angle scattering in the parabolic approximation, the other two wind components perpendicular to the direction of the incident wave do not contribute to the refractive index. Using again the parabolic equation method leads to:

$$< \psi(\vec{r}) >= \psi_0(\vec{r}) \exp\left(-\frac{k^2}{2} A_n(0) z\right) \tag{20}$$

$$A_n(\vec{\rho}) = \int_{-\infty}^{+\infty} \Phi_{nn}(\vec{r}) dz, \quad \Phi_{nn}(\vec{\rho}, z) = < n(\vec{r}_0) n^*(\vec{r}_0 + \vec{r}) > \tag{21}$$

here the superscript (*) means the complex conjugate.

3.3 LMSP based on OHE

Now, the generalized LMSP is used to solve the Operator Helmholtz Equation. The basic physical ideas described in section 2 are not changed. Only the scattering operator $\widehat{S}$ in equation (6) will be replaced by $\widehat{S}_L$:

$$\widehat{S}_L(\vec{R}, \vec{r}') < \psi(\vec{r}') >_{,}= k^4 \int\int G(\vec{R}, \vec{r}) < \mathbf{L} G(\vec{r}, \vec{r}') \mathbf{L}' >< \psi(\vec{r}') > d^3\vec{r}' d^3\vec{r} \tag{22}$$

Because the differential operator $\mathbf{L}$ takes the derivative only with respect to the variable $\vec{r}$, and the operator $\mathbf{L}'$ only with respect to the variable $\vec{r}'$, the integrand in the above equation can be simplified further and is written in the following form:

$$< \mathbf{L} G(\vec{r}, \vec{r}') \mathbf{L}' >= 4 \mathbf{D} G < \epsilon^T(\vec{r}) \epsilon(\vec{r}') > \mathbf{D}'^T \tag{23}$$

where the superscript (T) means the transpose of matrix.

$$\epsilon(\vec{r}) = \left[\frac{T'}{2}, \; -\frac{1}{2k^2}\frac{\partial T'}{\partial x_i}, \; \frac{i}{k^3}\frac{\partial u_l'}{\partial x_i}, \; \frac{i}{k^3} u_l'\right] \tag{24}$$

$$\epsilon(\vec{r}') = \left[\frac{T'}{2}, \; -\frac{1}{2k^2}\frac{\partial T'}{\partial x_j'}, \; \frac{i}{k^3}\frac{\partial u_m'}{\partial x_j'}, \; \frac{i}{k^3} u_m'\right] \tag{25}$$

$$\mathbf{D} = \left[1, \; \frac{\partial}{\partial x_i}, \; \frac{\partial^2}{\partial x_l \partial x_i}, \; \frac{\partial^3}{\partial x_l \partial x_i \partial x_i}\right] \tag{26}$$

$$\mathbf{D}' = \left[1, \; \frac{\partial}{\partial x_j'}, \; \frac{\partial^2}{\partial x_m' \partial x_i'}, \; \frac{\partial^3}{\partial x_m' \partial x_j' \partial x_j'}\right] \tag{27}$$

For the homogeneous medium $< \epsilon^T(\vec{r}) \epsilon(\vec{r}') >= \mathbf{E}(\vec{r} - \vec{r}')$. It is obvious that $\mathbf{E}$ is a 4×4 matrix. Hence:

$$\widehat{S}_L(\vec{R}, \vec{r}') < \psi(\vec{r}') >_{,}= 4k^4 \int\int G(\vec{R}, \vec{r}) \mathbf{D} G(\vec{r}, \vec{r}') \mathbf{E}(\vec{r} - \vec{r}') \mathbf{D}'^T < \psi(\vec{r}') > d^3\vec{r}' d^3\vec{r} \tag{28}$$

Like equation (8) the integral is a convolution product, too. The difference between this two equations is that the integrand of equation (28) contains two differential operators acting on the Green's function and the wave function, respectively. This makes our problem a little difficult.

However, Fourier transformation makes the derivation in $\vec{r}$-space a simple factor in $\vec{\kappa}$-space. After two dimensional FT equation (28) will be simplified and only z-integration is left. By extending the intergration limits to infinity (which is possible, because of the slabs being much thicker than correlation length), z-integration can be interpreted as a one-dimensional FT of function $\widetilde{\mathrm{E}}(\vec{\kappa}, z)$. After mathematical reduction the final result is similar to eq.(11):

$$< \widetilde{\psi}(\vec{\kappa}, z) > = e^{-\widetilde{S}_L(\vec{\kappa})z} < \widetilde{\psi}_0(\vec{\kappa}, z) > \tag{29}$$

$$\widetilde{S}_L(\vec{\kappa}) = \frac{1}{2}k^2 \int \frac{Q_- R}{aa'} \mathbf{K}^- \mathbf{\Phi}(\vec{\kappa}', a' - a) \mathbf{K}^T d^2\vec{\kappa}' \tag{30}$$

where

$$\mathbf{K}^- = [k, \kappa_1 - \kappa_1', \kappa_2 - \kappa_2', a'], \quad \mathbf{K} = [k, \kappa_1, \kappa_2, a] \tag{31}$$

$$Q_- = 1 + \frac{\mathbf{k}^- \cdot \mathbf{k}'}{k^2}, \quad R = 1 - \frac{\mathbf{k} \cdot \mathbf{k}'}{k^2}. \tag{32}$$

$$\mathbf{k}^- = (\kappa_1 - \kappa_1', \kappa_2 - \kappa_2', a'), \ \mathbf{k} = (\kappa_1, \kappa_2, a), \ \mathbf{k}' = (\kappa_1', \kappa_2', a' - a) \tag{33}$$

$\mathbf{\Phi}(\vec{\kappa}', a' - a)$: is the *3- dimensional* spectrum of the correlation function matrix.

$$\mathbf{\Phi} = \begin{bmatrix} \Phi_{00} & \Phi_{01} & \Phi_{02} & \Phi_{03} \\ \Phi_{10} & \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{20} & \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{30} & \Phi_{31} & \Phi_{32} & \Phi_{33} \end{bmatrix} \tag{34}$$

$$\Phi_{lm}(\vec{K}) = \int d^3\vec{r} e^{i\vec{K}\cdot\vec{r}} B_{lm}(\vec{r}), \quad l, m = 0, 1, 2, 3 \tag{35}$$

$$B_{00}(\vec{r}) = < \frac{1}{2}T'(\vec{r}')\frac{1}{2}T''(\vec{r}') > \qquad B_{0j}(\vec{r}) = < \frac{1}{2}T'(\vec{r}')u_j'(\vec{r}'') > \tag{36}$$

$$B_{j0}(\vec{r}) = < u_j'(\vec{r}')\frac{1}{2}T'(\vec{r}'') > \qquad B_{ij}(\vec{r}) = < u_i'(\vec{r}')u_j'(\vec{r}'') > \tag{37}$$

$$\vec{r} = \vec{r}' - \vec{r}'', \quad i, j = 1, 2, 3 \tag{38}$$

$< \widetilde{\psi}_0(\vec{\kappa}_\rho, z) >$ is the spectral form of the incident wave that propagates to z without scattering. The scattering function $\widetilde{S}_L$ differs from (12) in the factors Q_-, R, $\mathbf{K}^-$ and $\mathbf{K}$, which are caused by the refractive index operator. Furthermore the former spectrum function is extended to a spectrum matrix $\mathbf{\Phi}$, which contains all possible correlation functions of temperature and wind velocity field. Therefore, the assumption that the thickness of slabs being larger than the correlation length should be extended to: the slabs thickness must be larger than the maximal correlation length of all the 10 correlation lengths (because of the symmetry properties of matrix $\mathbf{\Phi}$, only 10 elements of this matrix are independent). If the incident wave is assumed to be a plane one and the parabolic approximation is used, the final result equation (29) can be further simplified and reduced to equation (20).

The results of this section can be regarded as the extention of those in section 2. If the wind velocity vanishes and only the temperature is left, equation (30) becomes:

$$\tilde{S}_L(\vec{\kappa}) = \frac{k^4}{2}\int \frac{Q_-R}{aa'}\Phi_{00}d^2\vec{\kappa} \qquad (39)$$

This correponds to equation (12) with: $\Phi_\mu = 4Q_-R\Phi_{00}$

4. CONCLUSIONS

A generalized form of the local method of small perturbations has been presented in this paper. Working directly from the Helmholtz equation instead of the parabolic equation the *small angle scattering* was replaced by *forward scattering*. Results for the first statistical moment of the wave function obtained by this method show corrections to the corresponding results with respect to the wave length - correlation length ratio. Furthermore, an Operator Helmholtz Equation was derived from Tatarskii's wave equation. Applying the above mentioned method to the Operator Helmholtz Equation leads to extended results including the wind velocity. Both generalizations of the Parabolic Equation Method are necessary for remote sensing the atmosphere's random temperature and wind field by sound measurements.

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REFERENCES

/1/ *Klug, H.,Sound speed profiles determined from outdoor sound propagation measurments.* Accepted for publication in J. Acoust. Soc. Am. (1991)

/2/ *Tatarskii, V.I., The effects of the turbulent atmosphere on wave propagation* Jerusalem 1971

/3/ *Clifford,S.F., The classical theory of wave propagation in a turbulent medium* in: *Strohbehn,J.W. Laser beam propagation in the atmosphere* Berlin 1978

/4/ *Strohbehn,J.W., Modern theories in the propagation of optical waves in a turbulent medium* in: *Strohbehn,J.W. Laser beam propagation in the atmosphere* Berlin 1978

/5/ *Große, R.,A local method of small perturbations based on Helmholtz equation. Part I: The first statistical moment.* Submitted to WAVES IN RANDOM MEDIA

/6/ *Candel, S.M., Numerical solution of wave scattering problems in the parabolic approximation,* J. Fluid. Mech. 90(3) 1979, 465-507.

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