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POWER SPECTRAL ESTIMATION USING ARMA MODELS

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ABSTRACT

All-pole modelling of spectra based on the use of Levinson's algorithm has several advantages including: the algorithm solves linear equations and thus is relatively fast; an error measure is produced at each iteration of the algorithm; the resulting estimate or model is minimum phase; the model reproduces the given data and results in high resolution estimates. These properties have made the Burg procedure for all-pole modelling a very useful and often used spectral estimator. The purpose of this paper is to present an Autogregressive-Moving Average (ARMA) model with the same kinds of properties as the Burg procedure. The intent is to develop an ARMA model that is more efficient in estimating certain classes of spectra, e.g., spectral envelops containing deep valleys.

Introduction

We shall present the theory and simulation results for an N pole, M zero ARMA (N,M) estimator that is based on the repeated use of Levinson's algorithm. As in the Burg procedure, an error measure is produced at each iteration stage. By monitoring this error one can obtain an indication of the order needed for the estimator. The resulting model is minimum phase and results in "high resolution" estimates of spectra.

ARMA models usually fall into one of two categories. The first class includes those which estimate the numerator and denominator coefficients simultaneously. This class of ARMA estimators results in a system of equations for the coefficients which are nonlinear. The second class

estimates the numerator and deonominator coefficients separately. These estimators are characterized by systems of linear equations.

The ARMA estimator presented here is based on linear prediction concepts. The pole-zero parameters are calculated separately and recursively. The method is similar to Burg's procedure [1] in that we can increase the model order of the numerator and denominator at each recursive step and monitor the improvement in the approximation. The algorithm allows one to calculate the numerator and denominator of the ARMA model in a variety of ways. For an ARMA (N,M) estimator we can initiate the process with, say, N = 1 and M = 0. This single-pole approximation is the first step in the algorithm. One can now add one or more zeros, followed by one or more poles. At any stage of the process one can also iterate holding the numerator and denominator orders constant. We have no theoretical results that define the best path to a given model order (N,M). Empirical results indicate that the addition of a single pole and zero at each iteration yields the best final ARMA model.

The essential calculation required in the algorithm is the computation of the autocorrelation coefficients of the polynomial $\frac{1}{p(z)}$ given the autocorrelation values of p(z). This problem cannot be solved with a finite algorithm. The remainder of the ARMA algorithm is a repeated use of Levinson's algorithm.

Description of the Algorithm

Let $|H(z)|^2$ be a power spectrum defined on $|z| = 1$. We desire to find an approximation of the form

$$\hat{H}_n(z) = g_n \frac{Q_m(z)}{P_n(z)} = g_n \frac{\sum_{i=0}^{m} q_i z^{-i}}{\sum_{i=0}^{n} p_i z^{-i}}, \quad p_0 = q_0 = 1$$

The process is initiated by choosing a constant f_1 and a first (or higher) order monic polynomial $P_1(z)$ via Levinson's algorithm so that

$|\frac{f_1}{P_1(z)}|$ is a single-pole approximation to $|H(z)|$. That is

$$|\hat{H}_1(z)| = |\frac{f_1}{P_1(z)}| \tilde{=} |H(z)|$$

If the approximation were exact, then

$$|\frac{f_1}{H(z)P_1(z)}| = 1$$

In general, $|\frac{f_1}{H(z)P_1(z)}|$ does not equal unity but rather some residual error. The algorithm continues by choosing a zero to match the residual error. Thus we choose a constant e_1 and monic first (or higher) order polynomial $Q_1(z)$ so that $|\frac{e_1}{Q_1(z)}|$ is an all-pole approximation to the residual $|\frac{1}{H(z)P_1(z)}|$. That is, $e_1/Q_1(z)$ is chosen so that

$$|\frac{Q_1(z)}{e_1}| \tilde{=} |\frac{1}{H(z)P_1(z)}|$$

At this point the approximation to the original spectrum is based on

$$|\hat{H}(z)| \tilde{=} |\frac{Q_1(z)}{e_1 P_1(z)}| = |H(z)|$$

The iteration process continues by choosing a second (or higher) order monic polynomial $P_2(z)$ to replace $P_1(z)$. Thus we calculate f_2 and $P_2(z)$ so that $|f_2/P_2(z)|$ is an all-pole model for $|\frac{H(z)}{Q_1(z)}|$. Then e_2 and $Q_2(z)$ are found to approximate $|\frac{1}{H(z)P_2(z)}|$.

We can summarize the algorithm as follows:

(1) Set $P_0(z) = Q_0(z) = 1$.

(2) Choose f_k and $P_k(z)$ so that $|\frac{f_k}{P_k(z)}|$ is a k-pole approximation to $|\frac{H(z)}{Q_{k-1}(z)}|$

(3) Choose e_k and $Q_k(z)$ so that $|\frac{e_k}{Q_k(z)}|$ is a k-pole approximation to $|\frac{1}{H(z)P_k(z)}|$. Then $\hat{H}_k(z) = \frac{Q_k(z)}{e_k P_k(z)}$.

We have outlined the algorithm in terms of polynomials in z. The calculations are actually performed in the sequence domain using the corresponding correlation sequences. The k-pole approximations in steps (1) and (2) are obtained by Levinson's algorithm. This implies we must have the correlation coefficients for $H(z)/Q_{k-1}(z)$ and $1/H(z)P_k(z)$. The primary computational problem is to obtain these correlation sequences. This calculation, in turn, reduces to finding the correlation sequence of a polynomial $\frac{1}{S(z)}$ given the correlation sequence of S(z). We call this the "inverse problem".

One method of solving the inverse problem is to use Levinson's algorithm to find an all-pole model 1/A(z) for S(z). Then A(z) is an approximation for 1/S(z). Now convolve the coefficients of A(z) to find a correlation sequence for 1/S(z).

Some Results

We have used the algorithm on a variety of spectra. In Figure 1 is shown a spectra characteristic with deep valleys and peaks. The ARMA method of the correct order matches the true spectrum reasonably well. In comparison an all-pole model of order M+N cannot approximate the deep valleys. The variance of the ARMA estimator based on overlaying several estimates is with 20% of the single realization shown.

In Figure 2 we used the algorithm to estimate sinusoids in additive white noise. These results are comparable to using the Burg procedure. The only advantage the ARMA procedure has is perhaps a little better definition at the very low signal-to-noise ratios.

Summary

We have developed an iterative, ARMA spectral estimator based on the repeated use of Levinson's algorithm to estimate both poles and zeros. The iterative nature of the model allows us to monitor the degree of approximation

and thus obtain some measure of order estimation. Good results are obtained for estimating spectra with deep spectral envelope valleys and peaks.

REFERENCES

[1] J. Makhoul, "Linear Prediction: A Tutorial Review", Proc. IEEE, Vol. 63, No. 4, 1975.

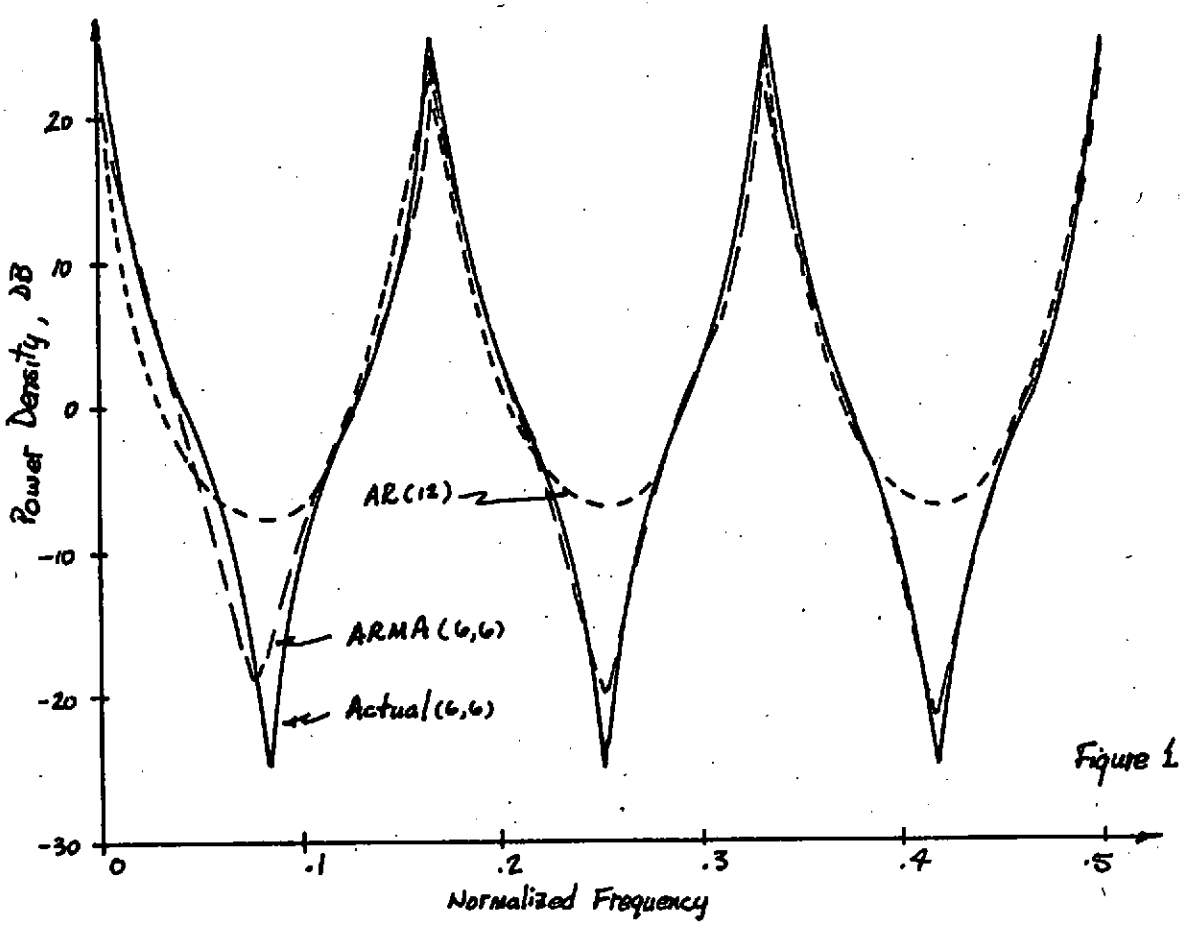

Figure 1

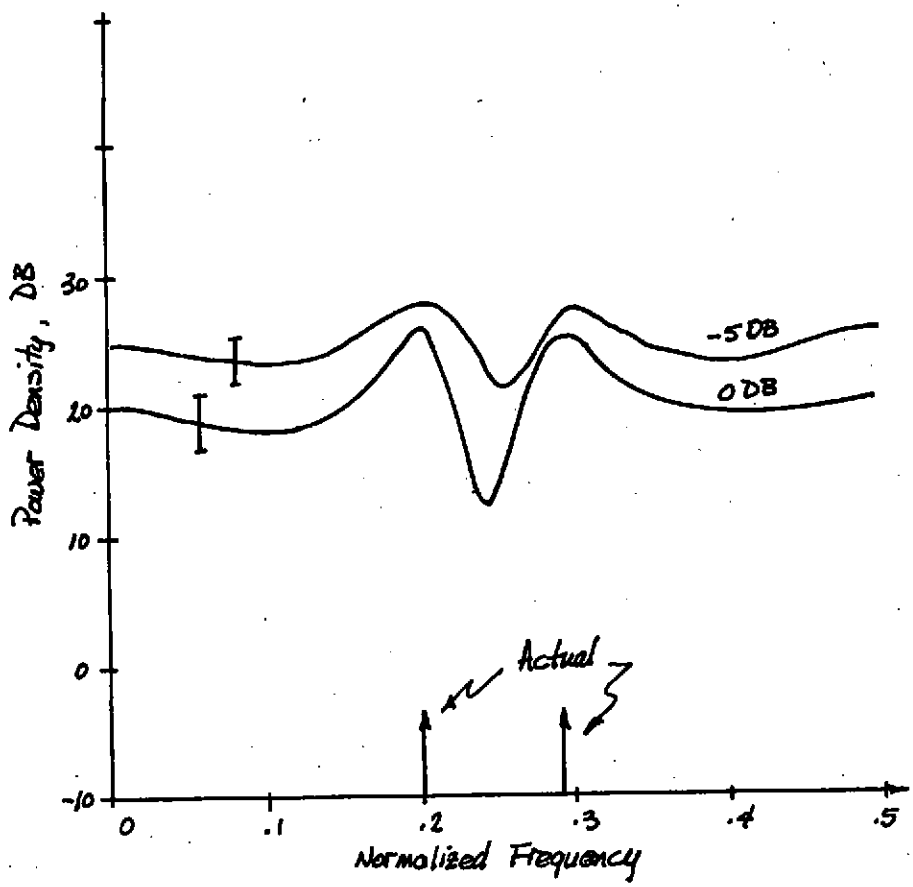
Power Density, DB
30
20
10
0
-10
-5 DB
0 DB
Actual
0
.1
.2
.3
.4
.5
Normalized Frequency

Figure 2