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VIBRATIONS AND WAVES IN LAMINATED ORTHOTROPIC CIRCULAR CYLINDERS

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Introduction

The natural vibrations and waves in elastic, homogeneous, isotropic circular cylinders, first studied by Pochhammer [1] and Chree [2], have since been investigated in considerable detail, as reviewed by Armenàkis, Gazis, and Herrmann [3] who give a set of tables of natural frequencies for various isotropic cylinders. The extension to layered cylinders has also received some attention, but, with the exception of axisymmetric motions in a two isotropic-layered cylinder, the behavior of general laminated orthotropic cylinders remains essentially unexplored. The frequency analysis of such bodies presents no conceptual difficulties; a frequency equation is generated by combining the solutions of the field equations for the various layers in accordance with the interlayer force and displacement continuity requirements and free surface conditions. However, for all but the simplest cases the algebra involved in the generation of the frequency equation is very cumbersome, and its solution is nearly intractable.

In this paper an extended Ritz technique is used to determine the natural frequencies and associated displacement and stress distributions of infinite circular cylinders. The cylinders are composed of an arbitrary number of bonded cylindrical layers, each composed of a distinct cylindrically orthotropic elastic material. The analysis is based upon the complete linear three-dimensional theory of elasticity. The essence of present technique, which was previously used for straight crested waves in laminated plates [4], is a mathematical idealization of the radial portion of the displacement behavior of the cylinder, while specifying explicitly the displacement form in the circumferential and axial directions. The idealization leads to an algebraic eigenvalue problem which is solved by an efficient direct-iterative eigenvalue solution technique [5]. For this paper the lowest ten frequencies and associated displacement distributions were obtained simultaneously so that stress distributions can easily be obtained. Several examples are presented which illustrate the accuracy and wide range of applicability of the method.

Formulation and Solution Method

To apply the extended Ritz technique the cylinder is divided into a number of cylindrical subregions with inner radius r_b and thickness h, with $h \ll r_b$. Each subregion is called a lamina and can possess distinct cylindrically orthotropic elastic properties and density. A lamina is not to be confused with a laminate, which can be modeled by a number of laminas. With reference to cylindrical coordinates (r,θ,z), the corresponding displacement components u_r, u_θ, u_z are taken in the form

$$
\begin{aligned}
u_r(r,\theta,z,t) &= U_r(r,t)\cos n\theta \cos(\pi z/\lambda) \\
u_\theta(r,\theta,z,t) &= U_\theta(r,t)\sin n\theta \cos(\pi z/\lambda) \\
u_z(r,\theta,z,t) &= U_z(r,t)\cos n\theta \sin(\pi z/\lambda)
\end{aligned}
\tag{1}
$$

where λ is the axial wave length, n is the circumferential mode number, and t denotes time. The functions U_r, U_θ, U_z are as yet undetermined. Although eqs. (1) represent a standing wave there is no loss in generality to travelling waves. The displacement form is the standard form which appears in studies of isotropic cylinder vibrations, and since it passes properly through displacement equations of motion and boundary conditions for cylindrically orthotropic materials, it is also applicable here. For this paper an approximate quadratic form in r is assumed for the variables $U_i(r,t)$, $i = r,\theta,z$,

$$
\begin{aligned}
U_j(r,t) &= U_{jb}(t)[1-3\hat{r}+2\hat{r}^2] + U_{ji}(t)[4\hat{r}-4\hat{r}^2] + \\
&+ U_{je}(t)[2\hat{r}^2-\hat{r}] \quad , \quad j = r,\theta,z
\end{aligned}
\tag{2}
$$

where $\hat{r}$ is a local radial ordinate for the lamina

$$
\hat{r} = (r - r_b)/h \quad ; \quad 0 \le \hat{r} \le 1 \tag{3}
$$

and U_{jb}, U_{ji}, U_{je} are generalized coordinates representing the displacement components at the back, middle, and front nodal surfaces, respectively.

The potential and kinetic energies for a particular lamina T and V, respectively, are obtained by integrating over a lamina volume of length λ

$$
\begin{aligned}
V = \frac{1}{2}\int_0^{\lambda}\int_0^{2\pi}\int_{r_b}^{r_b+h} &[C_{11}\epsilon_{rr}^2 + C_{22}\epsilon_{\theta\theta}^2 + C_{33}\epsilon_{zz}^2 + \\
&+ 2(C_{12}\epsilon_{rr}\epsilon_{\theta\theta} + C_{13}\epsilon_{rr}\epsilon_{zz} + C_{23}\epsilon_{\theta\theta}\epsilon_{zz}) + \\
&+ 4(C_{44}\epsilon_{rz}^2 + C_{55}\epsilon_{\theta z}^2 + C_{66}\epsilon_{r\theta}^2)] r\,dr\,d\theta\,dz
\end{aligned}
\tag{4}
$$

and

$$T = \frac{1}{2}\int_0^{\lambda}\int_0^{2\pi}\int_{r_i}^{r_i+h} \rho[\dot{U}_r^2 + \dot{U}_\theta^2 + \dot{U}_z^2]\, r\,dr\,d\theta\,dz$$

where ϵ_{ij} is the small strain tensor, C_{ij} are the elastic modulii and ρ is the mass density for the lamina, and the dot denotes differentiation in time. Substitution of the displacement forms, using appropriate strain displacement relations, into V and T gives in matrix notation

$$V = \frac{1}{2}\{r\}^T[k]\{r\}$$
$$T = \frac{1}{2}\{\dot{r}\}^T[m]\{\dot{r}\} \tag{5}$$

where [k] and [m] are the stiffness and mass matrices for the lamina and

$$\{r\}^T = \{U_{rb}, U_{\theta b}, U_{zb}, U_{ri}, U_{\theta i}, U_{zi}, U_{rc}, U_{\theta c}, U_{zc}\} \tag{6}$$

The difference L of the kinetic and potential energies of the complete cylinder of length λ is formed by summation of all the corresponding lamina quantities,

$$L = \frac{1}{2}\{\dot{U}\}^T[M]\{\dot{U}\} - \frac{1}{2}\{U\}^T[K]\{U\} \tag{7}$$

where [K] and [M] are the stiffness and mass matrices respectively and {U} is the set of generalized coordinates for the entire cylinder. Application of Hamilton's principle on the function L gives

$$[K]\{U\} + [M]\{\ddot{U}\} = 0 \tag{8}$$

For simple harmonic motion {U} is of the form

$$\{U\} = \{U_o\}e^{i\omega t} \tag{9}$$

where ω is the circular frequency. Substitution of eq. (9) into eq. (8) gives the eigenvalue problem

$$\{[K] - \omega^2[M]\}\{U_o\} = 0 \tag{10}$$

Equation (10) is examined by use of a direct iterative eigensolution technique [5] in which the rank of the problem is reduced by employing a limited number of generalized coordinates. A Stodola-Vionella type iteration on these generalized coordinates gives simultaneous convergence to the desired number of lowest frequencies. Thus it is possible to examine the general model, eqs. (10), without dealing directly with its eigensystem. In this paper eighteen coordinates were used to insure rapid convergence to the lowest ten frequencies. The procedure simultaneously gives modal displacement patterns so that stresses can be calculated directly using the constitutive relations.

Examples

A number of examples are presented to indicate the accuracy and range of applicability of the method. Several isotropic cylinders are examined and the results compared with those in [3]. A slightly orthotropic cylinder is investigated and the results compared with an approximate (six mode) elasticity theory, and two highly orthotropic examples are also presented. A final three layer orthotropic cylinder is considered to indicate the very complicated physical behavior of laminated orthotropic bodies. The examples give an indication of wide range of applicability of the method and also of its usefulness for determining the range of validity of various linear shell theories for laminated cylinders.

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